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CONTENTS

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A CLASS OF NUMBERS CONNECTED WITH PARTITIONS.

By E. T. Bell.

I. The Numbers A.

From their common origin in the elliptic theta and modular functions the theory of partitions and that of the representation of an integer as a sum of squares must be closely related. The innumerable relations thus suggested depend upon eight new systems of integers A which we define in § 1, express in terms of known arithmetical functions and calculate by simple recurrences in § 3. These numbers are functions of two integer parameters, the rank n and the degree r. The numbers of degrees 2, 3, 6, 9 are connected with the class number for binary quadratic forms of a negative determinant and, more generally, those of degrees r, $2r$ (r an arbitrary constant integer > 0) are related to the representation of an integer as a sum of r squares. The connection with partitions is effected through the concepts of the index and degree of a partition introduced in § 1. It will be seen that the subject, which is new, is of great extent.

§ 1. Consider all those partitions of the integer $n > 0$ in which no part appears more than r times. If in a particular one there are precisely a_j parts each of which occurs exactly j times, the r index of the partition is the hypercomplex number $(a_1, a_2, \cdots, a_r)$, and the partition is said to be of degree r.

The number of partitions of n having $(a_1, a_2, \cdots, a_r)$ as index will be written $A_n(a_1, a_2, \cdots, a_r)$, and if $x_1, x_2, \cdots, x_r$ are either constants or functions of a single parameter, we shall write

$$(1) \qquad A_n'(x_1, x_2, \cdots, x_r) \equiv \sum A_n(a_1, a_2, \cdots, a_r) x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r},$$

$\sum$ extending to all $(a_1, a_2, \cdots, a_r)$ for n constant.

When all the parts in each of the partitions enumerated by $A_n(a_1, a_2, \cdots, a_r)$ are restricted to be odd, O will be written in place of A,

$$(2) \qquad O_n'(x_1, x_2, \cdots, x_r) \equiv \sum O_n(a_1, a_2, \cdots, a_r) x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r}.$$

The functions (1), (2) have the conventional value 1 when $n = 0$.

Since in any partition of n no part can appear more than n times, n is the maximum degree of any partition of n. Hence when $r = n$ in (1), (2), there is no restriction upon the number of times that any part may occur in any of the indicated partitions.

Observing that the exponents n, $2n$, $3n$, $\cdots$ of q on the left of (3) can be written n, $n + n$, $n + n + n$, $\cdots$, we have

$$(3)\qquad \prod_{n=1}^{\infty}(1 + x_1q^n + x_2q^{2n} + \cdots + x_rq^{rn}) = \sum_{n=0}^{\infty} q^n A'_n(x_1, x_2, \cdots, x_r).$$

As always henceforth it is assumed that such a value of q has been chosen as to render absolutely convergent the product and series. It is easy to show that such a q exists in all cases treated in this paper. Similarly, Π extending to $m = 1, 3, 5, \cdots$, we have

$$(4)\qquad \prod (1 + x_1q^m + x_2q^{2m} + \cdots + x_rq^{rm}) = \sum_{n=0}^{\infty} q^n O'_n(x_1, x_2, \cdots, x_r).$$

There are two distinct divisions of the subject according as $x_1, x_2, \cdots, x_r$ are or are not numerical constants. Here we consider only the important cases in which

$$x_s = \binom{r}{s},\qquad x_s = (-1)^s\binom{r}{s}\qquad (s = 1, 2, \cdots, r),$$
$$x_s = \left\{ {r \atop s} \right\},\qquad x_s = (-1)^s\left\{ {r \atop s} \right\}\qquad (s = 1, 2, \cdots, n),$$

where $\binom{r}{s}$ is the binomial coefficient $r!/s!(r-s)!$, $\binom{0}{s} = \binom{0}{0} = 1$, and $\left\{ {r \atop s} \right\}$ is the rth figurate number of order s,

$$\left\{ {r \atop s} \right\} = \binom{r+s-1}{s} = \binom{s+r-1}{r-1}.$$

Let n, r denote positive integers. The first class of A numbers comprises the four systems

$$A_0(n, r) \equiv A'_n\left(-\binom{r}{1}, \binom{r}{2}, -\binom{r}{3}, \cdots, (-1)^r\binom{r}{r}\right),$$
$$A_1(n, r) \equiv A'_n\left(\binom{r}{1}, \binom{r}{2}, \binom{r}{3}, \cdots, \binom{r}{r}\right),$$
$$A_2(n, r) \equiv O'_n\left(\binom{r}{1}, \binom{r}{2}, \binom{r}{3}, \cdots, \binom{r}{r}\right),$$
$$A_3(n, r) \equiv O'_n\left(-\binom{r}{1}, \binom{r}{2}, -\binom{r}{3}, \cdots, (-1)^r\binom{r}{r}\right),$$

of rank n and degree r, and the second class is

$$A_0(n, -r) \equiv A'_n\left(\left\{ {r \atop 1} \right\}, \left\{ {r \atop 2} \right\}, \cdots, \left\{ {r \atop n} \right\}\right),$$
$$A_1(n, -r) \equiv A'_n\left(-\left\{ {r \atop 1} \right\}, \left\{ {r \atop 2} \right\}, \cdots, (-1)^n\left\{ {r \atop n} \right\}\right),$$
$$A_2(n, -r) \equiv O'_n\left(-\left\{ {r \atop 1} \right\}, \left\{ {r \atop 2} \right\}, \cdots, (-1)^n\left\{ {r \atop n} \right\}\right),$$
$$A_3(n, -r) \equiv O'_n\left(\left\{ {r \atop 1} \right\}, \left\{ {r \atop 2} \right\}, \cdots, \left\{ {r \atop n} \right\}\right),$$

of rank n and degree $-r$. These are definitions; neither set is obtained from the other by changing the sign of r. The reason for the notation appears in a moment. In each of the relevant partitions in $A_j(n, r)$ no part occurs more than r times, while in $A_j(n, -r)$ there is no such restriction.

The product extending to $n = 1, 2, 3, \cdots$, or to $m = 1, 3, 5, \cdots$, we

write in the usual notation

$$(5)\qquad q_0 = \prod (1 - q^{2n}), \qquad q_1 = \prod (1 + q^{2n}), \qquad q_2 = \prod (1 + q^m),$$
$$q_3 = \prod (1 - q^m),$$

where $q_j \equiv q_j(q)$, and see by (3), (4) that

$$(6)\qquad q_j^{\pm r} = \sum q^{2n} A_j(n, \pm r) \qquad (j = 0, 1),$$
$$(7)\qquad q_j^{\pm r} = \sum q^{n} A_j(n, \pm r) \qquad (j = 2, 3),$$

$\sum$ extending to $n = 0, 1, 2, \cdots$, and the upper signs or the lower being taken throughout. With (5) we have the well-known identity

$$(8)\qquad q_1 q_2 q_3 = q_1(\sqrt{q}) q_3 = 1,$$

from which and (5) we get

$$(9)\qquad A_1(n, s) = (-1)^n A_2(n, -s) = A_3(n, -s) \qquad (s \gtrless 0).$$

We may therefore confine our attention to methods for computing

$$(10)\qquad A_j(n, r) \qquad (j = 0, 1).$$

For easy reference we collect here the expansions of $\vartheta_j = \vartheta_j(q)$:

$$(11)\quad \vartheta_0(-q) = \vartheta_3 = \sum q^{n^2}, \quad \vartheta_1'(q^4) = \sum (-1|m) m q^{m^2}, \quad \vartheta_2(q^4) = \sum q^{m^2},$$

$\sum$ extending to $n = 0, \pm 1, \pm 2, \cdots$, $m = \pm 1, \pm 3, \pm 5, \cdots$, and $(a|b)$ being the Legendre-Jacobi symbol. We have also

$$(12)\qquad \vartheta_0 = q_0 q_3^2, \qquad \vartheta_1' = 2\sqrt[4]{q}\, q_0^3, \qquad \vartheta_2 = 2\sqrt[4]{q}\, q_0 q_1^2, \qquad \vartheta_3 = q_0 q_2^2,$$

whence, solving for q_j, we get

$$(13)\quad q_0^3 = \vartheta_1'/2\sqrt[4]{q}, \; q_2^6 = 2\sqrt[4]{q}\, \vartheta_3^2/\vartheta_0\vartheta_2, \; q_1^6 = \vartheta_2^2/4\sqrt{q}\, \vartheta_0\vartheta_3, \; q_3^6 = 2\sqrt[4]{q}\, \vartheta_0^2/\vartheta_2\vartheta_3,$$

and therefore by the transformation of order 2,

$$(14)\qquad q_1^3 = \vartheta_2/2\sqrt[4]{q}\, \vartheta_0(q^2), \qquad q_3^3 = 2\sqrt[8]{q}\, \vartheta_0/\vartheta_2(\sqrt{q}).$$

§ 2. To illustrate the definitions we verify for $n = 5, 6$ a result proved in § 5:

$$\sum (-1)^{a_1+a_3} 3^{a_1+a_2} A_n(a_1, a_2, a_3) = 0 \quad \text{or} \quad (-1|m)m$$

according as n is not or is $(m^2 - 1)/8$ where $m > 0$ is odd. When $n = 5$ the value of the right is 0, since 41 is not a square; when $n = 6$ we have $m = 7$ and the value of the right is -7. All the partitions of 5 are 5, 41, 32, 311, 221, 2111, 11111, the last of which is of degree 5. The 3 indices of the rest are respectively (1, 0, 0), (2, 0, 0), (2, 0, 0), (1, 1, 0), (1, 1, 0), (1, 0, 1), so that $A_5(2, 0, 0) = A_5(1, 1, 0) = 2$, $A_5(1, 0, 0) = A_5(1, 0, 1) = 1$. Substituting these in the left of the above relation, we find $-3 + 18 - 18$

$+ 3 = 0$. All the partitions of 6 are 6, 51, 42, 411, 33, 321, 3111, 222, 2211, 21111, 111111, of which the last two are of degrees 4, 6, and the 3 indices of the rest are respectively (1, 0, 0), (2, 0, 0), (2, 0, 0), (1, 1, 0), (0, 1, 0), (3, 0, 0), (1, 0, 1), (0, 0, 1), (0, 2, 0). Hence $A_6(2, 0, 0) = 2$, $A_6(a_1, a_2, a_3) = 1$ for each of the others. Proceeding as before we find $-3 + 18 - 9 + 3 - 27 + 3 - 1 + 9 = -7$. The sum on the left of the given identity is $A_0(n, 3)$.

§ 3. Denote by $\zeta_s(n)$, $\zeta'_s(n)$ the sum of the sth powers of all, of the odd, divisors of n. Taking logarithmic derivatives of (6) we find

$$nA_0(n, \pm r) = \mp r \sum \zeta_1(s) A_0(n - s, \pm r), \tag{15}$$

$$nA_1(n, \pm r) = \pm r \sum \zeta'_1(s) A_1(n - s, \pm r), \tag{16}$$

in which $\sum$ (as always henceforth in sums involving s under the sign) extends to all values of $s \geqq 0$, rendering the first arguments in the double argument functions positive. From (15), (16) we can write down the forms of the functions (10) as determinants involving ζ_1, ζ'_1. It is unnecessary to transcribe the results as the recurrences (15), (16), together with (19) to (21) offer a more practicable method of computation.

From $q_j^r q_j^{-r} = 1$ we have

$$A_j(n, -r) = - \sum A_j(s, r) A_j(n - s, -r), \tag{17}$$

and hence the explicit form of the A_j of negative degree in terms of the corresponding A_j of positive degree. We omit the determinant forms. It is therefore sufficient to discuss only the computation of

$$A_j(n, r) \qquad (j = 0, 1), \tag{18}$$

although (15), (16) are at least as useful as (17).

To eliminate all tentative processes from the calculation of (18) we must find recurrences for ζ_1, ζ'_1 in (15), (16). These can be found in many ways; the following is simple. Write $\theta_1(n) \equiv \zeta_1(\tfrac{1}{2}n) + \zeta'_1(\tfrac{1}{2}n)$ or $\zeta_1(n)$ according as n is even or odd, and $\eta_1(n) \equiv 2\zeta'_1(n) - \zeta_1(n)$ Then after some easy reductions of the logarithmic derivatives of the first and third of (12) we find

$$\theta_1(n) - 2\theta_1(n - 1^2) + 2\theta_2(n - 2^2) - \cdots = (-1)^{n-1} n\epsilon(n), \tag{19}$$

where $\epsilon(n) = 0$ or 1 according as n is not or is the square of a positive integer, and

$$\eta_1(n) + \eta_1(n - 1) + \eta_1(n - 3) + \cdots = n\epsilon(8n + 1), \tag{20}$$

the numbers 1, 3, $\cdots$ being triangular. From (19), (20) we compute the

$\theta_1(n)$, $\eta_1(n)$ by rapid recurrences, and hence $\zeta_1(n)$, $\zeta_1'(n)$ from

$$(21)\qquad \zeta_1(n) = \tfrac{1}{3}[2\theta_1(2n) - \eta_1(n)], \qquad \zeta_1'(n) = \tfrac{1}{3}[\theta_1(2n) + \eta_1(n)].$$

The computation of the A numbers has therefore been effected nontentatively.

II. RELATIONS WITH CLASS NUMBERS.

§ 4. A few will be sufficient. With the usual conventions* let $F(n)$, $F_1(n)$ denote the number of odd, even classes of binary quadratic forms of negative determinant $-n$, and write $D(n) = F(n) - F_1(n)$. Then there are the classical expansions

$$(22)\qquad \begin{aligned} &\vartheta_3^3 = 12\sum q^n D(n), \qquad \vartheta_2^3(q^4) = 8\sum q^{8n+3}F(8n+3),\\ &\vartheta_2(q^4)\vartheta_3^2(q^4) = 4\sum q^{4n+1}F(4n+1),\\ &\vartheta_3(q^2)\vartheta_2^2(q^2) = 4\sum q^{2n+1}F(4n+2), \end{aligned}$$

$\sum$ extending to $n = 0, 1, 2, \cdots$. From (8), (12) we find

$$(23)\qquad 2\sqrt[4]{q}\,\vartheta_3^3 = \vartheta_1' q_2^6, \quad \vartheta_2^3 = 4\sqrt{q}\,\vartheta_1' q_1^6, \quad q_3^2\vartheta_2\vartheta_3^2 = \vartheta_1' q_2^2, \quad q_3^2\vartheta_3\vartheta_2^2 = 2\sqrt[4]{q}\,\vartheta_1' q_1^2.$$

Translating these as they stand we get

$$(24)\qquad 12D(n) = \sum (-1)^s(2s+1)A_2(n - s - s^2, 6),$$

$$(25)\qquad F(8n+3) = \sum (-1)^s(2s+1)A_1(n - \tfrac{1}{2}s - \tfrac{1}{2}s^2, 6),$$

$$(26)\qquad 2\sum (-1)^s A_2(s, 2)F(4n + 1 - 4s) = \sum (-1)^s(2s+1)A_2(n - s - s^2, 2),$$

$$(27)\qquad \sum (-1)^s A_2(s, 2)F(8n + 2 - 4s) = \sum (-1)^s(2s+1)A_1(n - \tfrac{1}{2}s - \tfrac{1}{2}s^2, 2),$$

$$(27.1)\qquad \sum (-1)^s A_2(s, 2)F(8n + 6 - 4s) = 0;$$

while if we divide the first and second of (23) by q_2^6, q_1^6 respectively and apply (9) to the first result we find

$$(28)\qquad \sum (-1)^s A_1(s, 6)D(n - s) = 0 \quad \text{or} \quad m(-1\,|\,m),$$

$$(29)\qquad \sum A_1(n - 2s, -6)F(8s + 3) = 0 \quad \text{or} \quad m(-1\,|\,m)$$

according as n is not or is $(m^2 - 1)/4$ where $m > 0$ is odd.

We notice from $\vartheta_3^3 = q_0^3 q_2^6$, $\vartheta_2^3 = 8q_0^3 q_1^6 q^{3/4}$ the following, easily seen to be the same as (24), (25),

$$(30)\qquad 12D(n) = \sum A_0(s, 3)A_2(n - 2s, 6),$$

$$(31)\qquad F(8n+3) = \sum A_0(s, 3)A_1(n - s, 6).$$

In any of the above the A_2 can be replaced by their A_1 equivalents by means of (9). See also (37), (38), (49) to (54).

* H. J. S. Smith, "Report on the Theory of Numbers," Art. 135.

III. Relations with Square Functions.

§ 5. The number of such relations may be multiplied indefinitely. We therefore give only a few representative specimens.

Denote by $N(n, r)$ the number of representations of n as a sum of r squares whose roots are $\gtreqless 0$, and by $M(n, r)$ the like number in which all the squares are odd with roots > 0. Then $M(n, r) = 0$ if $n \not\equiv r \bmod 8$, and we have

(32) $\vartheta_3^r = \sum q^n N(n, r), \qquad \vartheta_2^r(q^4) = 2^r \sum q^{8n+r} M(8n + r, r),$

$\sum$ extending to $n = 0, 1, 2, \cdots$, with the convention that $N(0, r) = 1$.

Interpreted as it stands, the first of the identities (13) gives the result in § 2. Raise each of (14) to the rth power and use (32):

(33) $\sum (-1)^s A_1(s, 3r) N(n - s, r) = (-1)^n M(8n + r, r),$
(34) $\sum A_3(s, 3r) M(8n + r - 8s, r) = (-1)^n N(n, r).$

Raising the last two of (12) to the rth power and taking logarithmic derivatives of the results, we obtain recurrences for M, N:

(35) $nM(8n + r, r) = r \sum \eta_1(s) M(8n + r - 8s, r),$
(36) $nN(n, r) = -2r \sum (-1)^s \theta_1(s) N(n - s, r),$

with which (19), (20) are to be used. These may be solved to give M, N explicitly in terms of η_1, θ_1 if desired.

The special cases $r = 3$ of (33), (34) may be noticed. By (22) we have

(37) $12 \sum (-1)^s A_1(s, 9) D(n - s) = (-1)^s F(8n + 3),$
(38) $\sum A_3(s, 9) F(8n + 3 - 8s) = 12(-1)^n D(n).$

Another important square function of frequent occurrence is the following. Let $R(n, r, t)$ denote the number of representations of n as a sum of r squares of which precisely t are odd with roots > 0 and occupy the first t places, the roots of the $r - t$ even squares being $\gtreqless 0$ and not fixed as to order. Then

(39) $\vartheta_2^r(q^4) \vartheta_3^t(q^4) \doteq 2^r \sum q^{4n+r} R(4n + r, r + t, r).$

If wished, R can be easily expressed in terms of the corresponding function in which either or both of the signs and positions of the odd squares are free. From (39) or the definitions,

(40) $R(4n, t, 0) = N(n, t), \qquad R(8n + r, r, r) = M(8n + r, r);$

while from the last two of (12),

(41) $\sum (-1)^s A_1(s, 2t) R(8n + r - 4s, r + t, r)$
$= \sum A_0(s, r + t) A_1(n - s, 2r),$

from which, observing that $A_j(0, 0) = 1$, $A_j(n, 0) = 0 (n > 0)$, we have by (40),

$$M(8n + r, r) = \sum A_0(s, r)A_1(n - s, 2r), \tag{42}$$

$$\sum (-1)^s A_1(s, 2r)N(2n - s, r) = A_0(n, r); \tag{43}$$

while again from the last two of (12),

$$\sum (-1)^s A_2(s, 2r)R(4n - 8s + r, r + t, r) = \sum A_0(s, r + t)A_2(n - 2s, 2t), \tag{44}$$

whence by (40),

$$N(n, r) = \sum A_0(s, r)A_2(n - 2s, 2r), \tag{45}$$

$$\sum (-1)^s A_2(s, 2r) = M(8n + r - 8s, r) = A_0(n, r). \tag{46}$$

Some special cases of the above are of particular interest. From (22), (39) we have

$$R(4n + 1, 3, 1) = 4F(4n + 1), \qquad R(4n + 2, 3, 2) = 4F(4n + 2), \tag{47}$$

and therefore on putting $(r, t) = (1, 2), (2, 1)$ in (41), (44) and using the result that

$$A_0(n, 3) = 0 \quad \text{or} \quad m(-1|m) \tag{48}$$

according as n is not or is $(m^2 - 1)/8$ where $m > 0$ is odd, which follows from the first of (13), we find the remarkable relations

$$4\sum (-1)^s A_1(s, 4)F(8n + 1 - 4s) = \sum (-1)^s(2s + 1)A_1(n - \tfrac{1}{2}s - \tfrac{1}{2}s^2, 2), \tag{49}$$

$$4\sum (-1)^s A_1(s, 2)F(8n + 2 - 4s) = \sum (-1)^s(2s + 1)A_1(n - \tfrac{1}{2}s - \tfrac{1}{2}s^2, 4), \tag{50}$$

$$4\sum (-1)^s A_2(s, 2)F(4n + 1 - 8s) = \sum (-1)^s(2s + 1)A_2(n - s - s^2, 4), \tag{51}$$

$$4\sum (-1)^s A_2(s, 4)F(4n + 2 - 8s) = \sum (-1)^s(2s + 1)A_2(n - s - s^2, 2); \tag{52}$$

while in a similar way (42), (45) are seen to be the generalizations of (31), (30) respectively, and from (43), (46) we get*

$$12\sum (-1)^s A_1(s, 6)D(2n - s) = 0 \quad \text{or} \quad m(-1|m), \tag{53}$$

$$\sum (-1)^s A_2(s, 6)F(8n + 3 - 8s) = 0 \quad \text{or} \quad m(-1|m) \tag{54}$$

according as n is not or is $(m^2 - 1)/8$ where $m > 0$ is odd, and (30), (31) give (24), (25).

* The left of (53) is not an integral multiple of 12 since $D(0) = \frac{1}{12}$ by the usual conventions.

§ 6. Another special type deserves notice because most probably it contains only a finite number of distinct relations. It is well known that the number of representations of n as a sum of 2, 4, 6, or 8 squares, and the like for $8n + r$ ($r = 2, 4, 6, 8$) when all the squares are odd, can be expressed in terms of the real divisors of n alone. For certain special forms of n the same holds also for 10, 12 squares, but it is not at present definitely settled, although certain considerations give a strong presumption in favor of its probability, whether these exhaust all such cases. Again it is well known, being implicit in the *Fundamenta Nova* of Jacobi, that $R(4n + r, r + t, r)$ is also so expressible when $(r, t) = (1, 1), (2, 2), (2, 4), (4, 2), (3, 3), (4, 4)$. The appropriate functions of the divisors in all of the above cases are given in convenient form in (among other places) a former paper.* Using these in conjunction with the formulas involving M, N, R, we can readily write out the system of relations between numbers A and functions of divisors. To save space we omit the results.

§ 7. Thus far we have used only the simplest identities from the rudiments of the elliptic theta functions as the point of departure for obtaining relations between the numbers A and square functions, and the ease with which a profusion of results can be so obtained is sufficient evidence of the extent of the theory. We must, however, allude to two further sources of relations, each of which is incomparably more prolific than that which we have used. The first is the theory of transformation and the related modular equations, examples of which are contained in the second, viz., Jacobi's memoir† on infinite series in which the exponents are contained simultaneously in two different quadratic forms. Not attempting here an exhaustive analysis of the relations deduced from Jacobi's expansions, we shall conclude with two examples. In Jacobi's formulas $\sum$ refers to all integers i, k from $-\infty$ to ∞.

As a first example consider (loc. cit., p. 238) Jacobi's result which can be written

$$q^{10}q_0(q^{20})q_0(q^{100}) = \sum (-1)^{i+k} q^{(10i+3)^2+(10k+1)^2},$$

and denote by $U(n, r)$ the sum $\sum(-1)^{i_j+k_j}$ taken over all solutions of $n = \sum_1^r[(10i_j + 3)^2 + (10k_j + 1)^2]$, which we need not define verbally. Then proceeding as before we find

$$(55) \qquad \sum A_0(s, r)A_0(n - 5s, r) = U(40n + 10r, r),$$

$$(56) \qquad A_0(n, r) = \sum A_0(s, -r)U(40n + 10r - 200s, r),$$

$$(57) \qquad A_0(n, r) = \sum A_0(s, -r)U(200n + 10r - 40s, r).$$

* AMERICAN JOURNAL OF MATHEMATICS, Vol. 42 (1920), p. 168.

† Werke, Vol. 2, pp. 219–288.

When $r = 1$, we have by Euler's theorem $A_0(n, 1) = 0$ or $(-1)^a$ according as n is not or is $(3a^2 + a)/2$ where $a \gtreqless 0$, and evidently $A_0(n, -1)$ is the total number $P(n)$ of partitions of n, also $U(n, 1)$ is the excess of the total number of representations of n in the pair of forms

$$(20i + 3)^2 + (20k + 1)^2, \quad (20i + 13)^2 + (20k + 11)^2 \quad (i, k \gtreqless 0)$$

over the like number for the pair

$$(20i + 13)^2 + (20k + 1)^2, \quad (20i + 3)^2 + (20k + 11)^2 \quad (i, k \gtreqless 0),$$

and from (55) this excess is equal to the excess of the total number of representations of $24n + 6$ in the pair of forms

$$(12i + 1)^2 + (12k + 1)^2, \quad (12i + 7)^2 + (12k + 7)^2 \quad (i, k \gtreqless 0)$$

over the like number for the pair

$$(12i + 7)^2 + (12k + 1)^2, \quad (12i + 1)^2 + (12k + 7)^2 \quad (i, k \gtreqless 0);$$

while from (56), (57) the value of either sum

$$\sum P(s)U(40n + 10 - 200s, 1), \qquad \sum P(s)U(200n + 10 - 40s, 1)$$

is 0 or $(-1)^a$ according as n is not or is $(3a^2 + a)/2$, $a \gtreqless 0$. Conclusions somewhat similar to the first of these can be read off by (48) from (55).

From the modular equation for the transformation of order 7, Jacobi obtains (loc. cit., p. 288) a result equivalent to

$$q^{17}q_2(q^{24})q_0(q^{168}) = q_0^{-1}(q^{48}) \sum (-1)^k q^{3(4i+1)^2+14(6k+1)^2};$$

whence we find

$$(58) \quad \sum A_0(s, r)A_2(n - 18s, r) = \sum A_0(s, -r)E(24n + 17 - 96s, r),$$

in which $E(n, r)$ is the excess of the number of representations of n in the form

$$\sum_{i=1}^{r} [3(4a_i + 1)^2 + 14(12b_i + 1)^2] \qquad (a_i, b_i \gtreqless 0)$$

over the like number for

$$\sum_{i=1}^{r} [3(4a_i + 1)^2 + 14(12b_i + 7)^2] \qquad (a_i, b_i \gtreqless 0).$$

When $r = 1$, $E(n, 1)$ is the excess of the number of representations of n in the first of the following forms over the like for the second,

$$3(4a + 1)^2 + 14(12b + 1)^2, \qquad 3(4a + 1)^2 + 14(12b + 7)^2 \qquad (a, b \gtreqless 0),$$

and we have

$$(59) \quad \sum (-1)^a Q(n - 9a - 27a^2) = \sum P(s)E(24n + 17 - 96s, 1),$$

where $Q(n)$ is the number of partitions of n into distinct odd parts and the sum on the left refers to all $a \gtreqless 0$ rendering $n - 9a - 27a^2$ positive.

It is a feature of this subject that all indicated computations can be performed non-tentatively. In particular, all of the arithmetical functions occurring can be calculated by recurrence. For example, taking logarithmic derivatives of the rth power of Jacobi's identity, we find for $E(n, r)$ the recurrence

$$(60)\quad nE(24n + 17r, r) = - \sum \lambda_1(s + 1)E(24n + 17r - 24s - 24, r),$$

where $\lambda_1(n) = - \zeta_1'(n)$ if $n \equiv j \bmod 28$, $j \neq 0, 14$;

$$n \equiv 0 \bmod 28, \qquad \lambda_1(n) = \zeta_1'(n) + 4\zeta_1\left(\frac{n}{4}\right) + 14\zeta_1\left(\frac{n}{14}\right),$$

$$n \equiv 14 \bmod 28, \qquad \lambda_1(n) = \zeta_1'(n) + 14\zeta_1\left(\frac{n}{4}\right),$$

and ζ_1, ζ_1' are to be calculated as in (19)–(21).

December, 1922.

NOTE ON A NEW TYPE OF SUMMABILITY.

By Norbert Wiener.

We have

$$\frac{x^2}{4}-\frac{\pi x}{2}+\frac{\pi^2}{6}=\cos x+\frac{\cos 2x}{4}+\cdots+\frac{\cos nx}{n^2}+\cdots. \tag{1}$$

From this it may be deduced at once that

$$\frac{\pi x - x^2}{2}=\sin^2 x+\frac{\sin^2 2x}{4}+\cdots+\frac{\sin^2 nx}{n^2}+\cdots, \tag{2}$$

whence

$$1-\frac{1}{\pi n}=\frac{2n}{\pi}\left\{\sin^2\frac{1}{n}+\frac{1}{4}\sin^2\frac{2}{n}+\cdots+\frac{1}{k^2}\sin^2\frac{k}{n}+\cdots\right\}. \tag{3}$$

Moreover,

$$\lim_{n\to\infty}\frac{2n}{\pi k^2}\sin^2\frac{k}{n}=0. \tag{4}$$

This suggests that given a series $\sum_1^\infty a_n$, we may consider

$$T(a_n)=\lim_{n\to\infty}\frac{2n}{\pi}\left\{a_1\sin^2\frac{1}{n}+\frac{a_1+a_2}{4}\sin^2\frac{2}{n}+\cdots+\frac{a_1+\cdots+a_k}{k^2}\sin^2\frac{k}{n}+\cdots\right\} \tag{5}$$

as a generalized sum of the a_n's. This definition is of the sort called linear and regular by Carmichael* and Hurwitz,† since (3) and (4) hold, and series (3) is a series of positive terms. Hence it evaluates every convergent series correctly.

Let us now apply our method of summation to the Fourier series

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx+\sum_1^\infty\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos n(x-x')dx. \tag{6}$$

The nth partial sum of this is, as is well known,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\frac{\sin(2n+1)\dfrac{x-x'}{2}}{\sin\dfrac{x-x'}{2}}dx. \tag{7}$$

* *Bull. Am. Math. Soc.*, Vol. 25 (1918–19), p. 97.

† Ibid., Vol. 28 (1922), p. 20.

Hence we get for (5)

$$\lim_{n\to\infty} \frac{2n}{\pi} \sum_{1}^{\infty} \frac{1}{2\pi k^2} \sin^2 \frac{k}{n} \int_{-\pi}^{\pi} f(x) \frac{\sin (2n+1) \dfrac{x-x'}{2}}{\sin \dfrac{x-x'}{2}} dx. \tag{8}$$

This leads us to consider the series

$$\begin{aligned}
&\frac{2n}{\pi} \sum_{k=1}^{\infty} \frac{1}{2\pi k^2} \sin^2 \frac{k}{n} \frac{\sin (2k+1)u}{\sin u} \\
&= \frac{2n}{\pi} \sum_{k=1}^{\infty} \frac{1}{4\pi k^2} \left(1 - \cos \frac{2k}{n}\right) (\sin 2ku \cot u + \cos 2ku) \\
&= \frac{2n}{\pi} \sum_{k=1}^{\infty} \frac{1}{4\pi k^2} \left\{ \cot u \left[\sin 2ku - \sin 2ku \cos \frac{2k}{n} \right] \right. \\
&\qquad\qquad \left. + \cos 2ku - \cos 2ku \cos \frac{2k}{n} \right\} \\
&= \frac{2n}{\pi} \sum_{k=1}^{\infty} \frac{1}{4\pi k^2} \left\{ \cot u \left[\sin 2ku - \frac{1}{2} \sin 2k \left(u + \frac{1}{n}\right) \right. \right. \\
&\qquad\qquad \left. - \frac{1}{2} \sin 2k \left(u - \frac{1}{n}\right) \right] \\
&\qquad\qquad \left. + \cos 2ku - \frac{1}{2} \cos 2k \left(u + \frac{1}{n}\right) - \frac{1}{2} \cos 2k \left(u - \frac{1}{n}\right) \right\} \\
&= \frac{n}{4\pi^2} \cot u [- 2 \int_0^u \log (4 \sin^2 x) dx \\
&\qquad + \int_0^{u+(1/n)} \log (4 \sin^2 x) dx + \int_0^{u-(1/n)} \log (4 \sin^2 x) dx] \\
&\qquad + \frac{n}{4\pi^2} \left[2 \left(u^2 - \pi u + \frac{\pi^2}{6}\right) - \left(\left(u + \frac{1}{n}\right)^2 \right. \right. \\
&\qquad \left. \left. - \pi \left(u + \frac{1}{n}\right) + \frac{\pi^2}{6} \right) - \left(\left(u - \frac{1}{n}\right)^2 - \pi \left(u - \frac{1}{n}\right) + \frac{\pi^2}{6} \right) \right] \\
&= \frac{n}{4\pi^2} \cot u [\int_u^{u+(1/n)} \log (4 \sin^2 x) dx \\
&\qquad\qquad + \int_u^{u-(1/n)} \log (4 \sin^2 x) dx] - \frac{1}{2\pi^2 n} \\
&= \frac{1}{4\pi^2} \cot u \log \frac{\sin^2 \xi_1}{\sin^2 \xi_2} - \frac{1}{2\pi^2 n},
\end{aligned} \tag{9}$$

where ξ_1 lies between u and $u + \frac{1}{n}$, ξ_2 between $u - \frac{1}{n}$ and u, and $0 < u < \pi$. As n increases, expression (9) converges uniformly to 0 over $(\epsilon, \pi - \epsilon)$. Moreover, in the neighborhood of $u = 0$, expression (9) is positive, while it never has a negative value less than $\frac{1}{2\pi^2 n}$. Series (9) is even in u.

The partial sums of series (9) are subject to the formal transformation

$$
\begin{aligned}
&\frac{2n}{\pi}\sum_{1}^{m}\frac{1}{2\pi k^2}\sin^2\frac{k}{n}\frac{\sin(2k+1)u}{\sin u}\\
&=\frac{2n}{\pi}\left\{\frac{1}{2\pi}\sin^2\frac{1}{n}+\frac{1}{\pi}\sin^2\frac{1}{n}\cos 2u\right.\\
&\quad+\frac{1}{8\pi}\sin^2\frac{2}{n}+\frac{1}{4\pi}\sin^2\frac{2}{n}\cos 2u+\frac{1}{4\pi}\sin^2\frac{2}{n}\cos 4u\\
&\quad+\frac{1}{18\pi}\sin^2\frac{3}{n}+\frac{1}{9\pi}\sin^2\frac{3}{n}\cos 2u\\
&\qquad\qquad\qquad+\frac{1}{9\pi}\sin^2\frac{3}{n}\cos 4u+\frac{1}{9\pi}\sin^2\frac{3}{n}\cos 6u \qquad (10)\\
&\quad+\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
&\quad+\frac{1}{2\pi m^2}\sin^2\frac{m}{n}+\frac{1}{\pi m^2}\sin^2\frac{m}{n}\cos 2u\\
&\qquad\qquad\qquad\left.+\cdots+\frac{1}{\pi m^2}\sin^2\frac{m}{n}\cos 2mu\right\}\\
&=\frac{2n}{\pi}\left\{\sum_{1}^{m}\frac{1}{2\pi k^2}\sin^2\frac{k}{n}+\cos 2u\sum_{1}^{m}\frac{1}{\pi k^2}\sin^2\frac{k}{n}\right.\\
&\qquad\left.+\cos 4u\sum_{2}^{m}\frac{1}{\pi k^2}\sin^2\frac{k}{n}+\cdots+\cos 2mu\frac{1}{\pi m^2}\sin^2\frac{m}{n}\right\}.
\end{aligned}
$$

This suggests an investigation of the series

$$\frac{2n}{\pi}\left\{\sum_{k=1}^{\infty}\frac{1}{2\pi k^2}\sin^2\frac{k}{n}+\sum_{j=1}^{\infty}\cos 2ju\sum_{k=j}^{\infty}\frac{1}{\pi k^2}\sin^2\frac{k}{n}\right\}. \qquad (11)$$

The sum of the squares of the coefficients of (11) may be shown to converge, since the mth coefficient is of the order of $1/m$. Since the coefficients of (10) are positive, less than those of (11) and convergent to those of (11), it follows that the sum of (10) converges in the mean to the sum of (11) with increasing m. Hence series (9) can be integrated term by term when multiplied by any summable function of summable square. Consequently if $f(x)$ is a summable function of summable square, we may write for (8)

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}f(x)\frac{2n}{\pi}\sum_{1}^{\infty}\frac{1}{2\pi k^2}\sin^2\frac{k}{n}\,\frac{\sin(2k+1)\dfrac{x-x'}{2}}{\sin\dfrac{x-x'}{2}}\,dx. \qquad (12)$$

Let us write (12) in the form $\lim_{n\to\infty}\int_{-\pi}^{\pi}f(x)G_n(x,x')dx$. We have already shown that over $(-\pi, x'-\epsilon)$ and $(x'+\epsilon, \pi)$,

$$\lim_{n\to\infty}G_n(x,x')=0 \qquad (13)$$

uniformly in x, and that $G_n(x, x') + \dfrac{1}{2\pi^2 n}$ is positive. Moreover,

$$\begin{aligned}\int_{-\pi}^{\pi} G_n(x, x')dx &= \int_{-\pi}^{\pi} \frac{2n}{\pi}\left\{ \sum_1^\infty \frac{1}{2\pi k^2}\sin^2\frac{k}{n}\right. \\ &\qquad \left. + \sum_{j=1}^\infty \cos j(x - x') \sum_{k=j}^\infty \frac{1}{\pi k^2}\sin^2\frac{k}{n}\right\} dx \\ &= \frac{2n}{\pi}\sum_1^\infty \frac{1}{k^2}\sin^2\frac{k}{n} \\ &= 1 - \frac{1}{\pi n}.\end{aligned} \tag{14}$$

From all these facts it follows that if $f(x)$ is continuous at $x = x'$,

$$\begin{aligned}\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)G_n(x, x')dx &= \lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\left[G_n(x, x') + \frac{1}{2\pi^2 n}\right]dx \\ &= \lim_{n\to\infty}\int_{-\pi}^{x'-\epsilon} f(x)\left[G_n(x, x') + \frac{1}{2\pi^2 n}\right]dx \\ &\quad + \lim_{n\to\infty}\int_{x'-\epsilon}^{x'+\epsilon} f(x)\left[G_n(x, x') + \frac{1}{2\pi^2 n}\right]dx \\ &\quad + \lim_{n\to\infty}\int_{x'+\epsilon}^{\pi} f(x)\left[G_n(x, x') + \frac{1}{2\pi^2 n}\right]dx \\ &= F\lim_{n\to\infty}\int_{x'-\epsilon}^{x'+\epsilon}\left[G_n(x, x') + \frac{1}{2\pi^2 n}\right]dx \\ &= F\lim_{n\to\infty}\int_{-\pi}^{\pi} G_n(x, x')dx = F,\end{aligned} \tag{15}$$

F being some quantity lying between the upper and the lower bounds of $f(x)$ over the interval $(x' - \epsilon, x' + \epsilon)$. Hence

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)G_n(x, x')dx = f(x'). \tag{16}$$

By an obvious modification of this proof, if $f(x' + 0)$ and $f(x' - 0)$ exist,

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)G_n(x, x')dx = \tfrac{1}{2}[f(x' + 0) + f(x' - 0)]. \tag{17}$$

In other words, at any point at which a summable function f of summable square is continuous, its Fourier series may be correctly summed by the method of this paper, and at any point at which f has determinate limits to the right and to the left, its Fourier series is summable to the mean of these values. Since a function is bounded in the neighborhood of a point of continuity, and since the contributions to the partial sums of a Fourier series at a point x' due to values of f for arguments outside $(x' - \epsilon, x' + \epsilon)$ converge uniformly to zero, the condition that f be of summable square is unessential.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
November 28, 1922.

ON MEDIATE CARDINALS.

By Dorothy Wrinch.

A "mediate cardinal" is defined in "Principia Mathematica" as a cardinal which is neither inductive nor reflexive and it is established $*(124\cdot61)$ that the multiplicative axiom implies the non-existence of mediate cardinals. The converse implication is not established, and there seems to be no reason to suppose it is true. The relation of the existence of mediate cardinals to the multiplicative axiom is therefore one-sided and offers a contrast to the mutual implications of the comparability of cardinals, the well-orderability of classes and the multiplicative axiom. In this paper it is proposed to investigate other classes of cardinals which are not Alephs, beyond the mediate cardinals of "Principia Mathematica," and instead of the one-sided implication between the multiplicative axiom and the non-existence of mediate cardinals to establish an equivalence between the axiom and the non-existence of certain cardinals which are not Alephs.

If we use the term "comparable" in such a sense that μ is comparable with ν when μ is greater than, equal to, or less than ν, a mediate cardinal in "Principia Mathematica" is a cardinal comparable with the inductive cardinals but not with $\mathbf{A}_0$.* We wish to discuss the nature of those cardinals (if there are any such) which are comparable with Alephs less than $\mathbf{A}_\xi$, but not with $\mathbf{A}_\xi$, for different values of ξ. In "Principia Mathematica" it is found that a part of the multiplicative axiom is sufficient to imply the non-existence of the non-inductive non-reflexive cardinals and instead of the proposition

$$\text{Mult Ax} \,.\, \supset \,.\, NC \text{ med} = \Lambda$$

there is the proposition $*124\cdot56$

$$\mathbf{A}_0 \,\epsilon\, NC \text{ mult} \,.\, \supset \,.\, NC \text{ med} = \Lambda.$$

It seems worth while to pursue the same course with the cardinals to be considered in this paper. For this purpose, it is necessary to establish certain propositions of little interest in themselves. This is done in $*1$.

$*1$.

κ and λ are similar classes of mutually exclusive similar classes. There exists therefore a correlator S of κ and λ, such that if $\alpha \,\epsilon\, \lambda$, $S\text{‘}\alpha \; sm \; \alpha$. We

* $\mathbf{A}$ has been substituted for the Hebrew Character Aleph usually employed.

will put

$$Crp(S)'\alpha = S'\alpha \; \overline{sm} \; \alpha, \qquad \text{Df.}$$

as in "Principia Mathematica," $*111\cdot02$. Then $Crp(S)'\alpha$ consists of the class of correlators between α and its S-correlate. It is established that under these circumstances

$$R \in \epsilon_\Delta{}'Crp(S)``\lambda \,.\supset.\, \dot{s}'D'R \in (s'\kappa)\overline{sm}(s'\lambda),$$

so that if an R can be found which consists of a selection from the various classes $Crp(S)'\alpha$ where $\alpha \in \lambda$, $\dot{s}'D'R$ is a correlator of $s'\kappa$ with $s'\lambda$. It follows that if a selection from the class of classes $Crp(S)``\lambda$ exists, $s'\kappa$ and $s'\lambda$ are similar. And this is certainly the case if $Nc'\lambda \in NC$ mult. Hence we get the proposition

$$*1\cdot1 \quad \vdash :. Nc'\lambda \in NC \text{ mult} \,.\supset:\, \kappa, \lambda \in Cls^2 \text{ excl}. \; \exists ! \; \kappa \overline{sm} \lambda \cap Rl'sm \,.\supset.\, s'k \; sm \; s'\lambda.$$

κ is a class consisting of $\aleph_\xi$ classes of mutually exclusive classes, each having $\aleph_\eta$ members. Let α be an $\aleph_\eta$ and β an $\aleph_\xi$. $\alpha \downarrow ; `` \beta$ and κ will then be similar classes of similar mutually exclusive classes. It follows from $1\cdot1$ that

$$Nc'\kappa \in NC \text{ mult} \,.\supset:\, \beta \in \aleph_\xi \,.\, \alpha \in \aleph_\eta \,.\, \kappa \in Cls^2 \text{ excl} \cap \aleph_\xi \cap Cl'\aleph_\eta \,.\supset.\, s'\kappa \; sm \; s'\alpha \downarrow ; `` \beta.$$

Since $s'\alpha \downarrow ; ``\beta = \beta \times \alpha$ and $Nc'(\alpha \times \beta) = Nc'(\beta \times \alpha)$ [P.M. $**113\cdot02 \cdot141$] we get the proposition

$$*1\cdot2 \quad \vdash :. \aleph_\xi \in NC \text{ mult} \,.\supset:\, \kappa \in Cls^2 \text{ excl} \cap \aleph_\xi \cap Cl'\aleph_\eta \,.\supset.\, s'\kappa \in \aleph_\eta \times \aleph_\xi.$$

μ and ν are similar classes of mutually exclusive classes. The cardinal number of any member of μ is less than or equal to the cardinal of any member of ν. Then $Nc's'\mu$ is less than or equal to $Nc's'\nu$.

For there exists an S correlating members of μ with members of ν. If $\beta \in \nu$, there exists a subclass of β having the same cardinal as $S'\beta$, the member of μ with which S correlates β. Putting

$$CN(S)'\beta = Cl'\beta \cap Nc'S'\beta, \qquad \text{Df.}$$

we have

$$\beta \in \nu \,.\supset.\, \exists ! CN'\beta.$$

And there are $Nc'\mu$ β's and $Nc'\mu$ $CN(S)'\beta$'s. If $Nc'\mu \in NC$ mult, a selection ν' can be taken from $CN``\nu$. Then

$$\exists ! \mu \; sm \; \nu' \cap Cl'sm.$$

Hence by $1 \cdot 1$, if $Nc`\mu \in NC$ mult

$$s`\mu \, \overline{sm} \, s`\nu'.$$

Since $s`\nu' \subset s`\nu$ it follows that $Nc`s`\mu \leq Nc`s`\nu$. Hence

$*1 \cdot 3 \quad \vdash :: Nc`\mu \in NC \text{ mult} . \supset :. \mu, \nu \in Cls^2 \text{ excl. } \mu sm \nu : \alpha \in \mu \cdot \beta \in \nu$
$\qquad . \supset_{\alpha\beta} . Nc`\alpha \leq Nc`\beta : \supset . Nc`s`\mu \leq Nc`s`\nu.$

It follows immediately from $1 \cdot 3$, $1 \cdot 2$, putting $\nu = \gamma \downarrow ; ``\delta$ where $\gamma \in \aleph_\eta$, $\delta \in \aleph_\xi$,

$*1 \cdot 4 \quad \vdash :: Nc`\mu \in NC \text{ mult} . \supset :. \mu \in Cls^2 \text{ excl.} \cap \aleph_\xi : \alpha \in \mu . \supset_\alpha . Nc`\alpha$
$\qquad < \aleph_\eta : \supset . Nc`s`\mu \leq \aleph_\xi \times \aleph_\eta.$

By $1 \cdot 3$, if $\aleph_\zeta \in NC$ mult, an $\aleph_\xi \times \aleph_\zeta$, where $\xi < \zeta$, is less than or equal to a class which is an $\aleph_\zeta \times \aleph_\zeta$, since it consists of $\aleph_\zeta$ classes all of which have less than $\aleph_\zeta$ members. And

$$\aleph_\zeta \times \aleph_\zeta = \aleph_\zeta. \tag{1·41}*$$

Hence

$$\aleph_\zeta \times \aleph_\xi \leq \aleph_\zeta. \qquad \text{[P.M. } *117 \cdot 6\text{]}$$

But

$$\aleph_\zeta \times \aleph_\xi \geq \aleph_\zeta.$$

Therefore

$$\aleph_\zeta \times \aleph_\xi = \aleph_\zeta,$$

which with $1 \cdot 41$ gives

$*1 \cdot 5 \qquad \vdash :. \xi \leq \zeta \cdot \aleph_\zeta \in NC \text{ mult} . \supset : \aleph_\zeta \times \aleph_\xi = \aleph_\zeta.$

κ is a class of mutually exclusive classes no two of which are similar. The cardinals of the members of κ are

$$\aleph_{\eta+a_0}, \quad \aleph_{\eta+a_1}, \quad \cdots \quad \aleph_{\eta+a_\zeta}, \quad \cdots, \qquad (\zeta < \omega_\xi)$$

where the Alephs form an ω_ξ-series, having $\aleph_z$ as limit. κ then consists of $\aleph_\xi$ classes each with less than $\aleph_z$ members. By $1 \cdot 4$

$$\aleph_\xi \in NC \text{ mult } . \supset . Nc`s`\kappa \leq \aleph_z \times \aleph_\xi.$$

By $1 \cdot 5$ since $\xi < z$

$$\aleph_z \in NC \text{ mult} . \supset . Nc`s`\kappa \leq \aleph_z.$$

Therefore if P is the series of Alephs in order of magnitude and $\Sigma_c`\lambda$ the arithmetic sum of a class of cardinals λ,

$*1 \cdot 6 \quad \vdash : \lambda \in Cl`(NC \cap C``\Omega) \cdot lt_P`\lambda \in NC \text{ mult.} \supset . \Sigma_c`\lambda \leq lt_P`\lambda.$

λ is a class of Alephs whose limit in order of magnitude is $\aleph_z$. Suppose

* See Jourdain, "The Multiplication of Alephs," *Mathematische Annalen*, Bd. LXV, pp. 506–512.

$(\exists x).\ \Sigma_c{}^{\backprime}\lambda = x\,.\,x < \aleph_z$. Since $\aleph_z = lt_P{}^{\backprime}\lambda$

$$(\exists \eta).\quad x < \eta.\quad \Sigma_c{}^{\backprime} \geq \eta.\quad \eta < \aleph_z.$$

Hence

$$(\exists x).\quad \Sigma_c{}^{\backprime}\lambda = x.\quad x < \aleph_z\,.\supset.\,\Sigma_c{}^{\backprime}\lambda > x.$$

Therefore $\sim (\exists x).\ \ x < \aleph_z.\ \ \Sigma_c{}^{\backprime}\lambda = x$. Hence with $1\cdot 6$ we get

$*1\cdot 7 \quad \vdash : \lambda \in Cl{}^{\backprime}(NC \cap C{}^{\backprime\backprime}\Omega).\ \ lt_P{}^{\backprime}\lambda \in NC \text{ mult}\,.\supset.\,\Sigma_c{}^{\backprime}\lambda = lt_P{}^{\backprime}\lambda.$

$*2.$

We will make use of the definition in "Principia Mathematica,"

$$\text{spec}{}^{\backprime}\mu = \hat{\nu}[\nu < \mu\,.\,v\,.\,\nu \geq \mu]. \qquad [\text{P.M. } *120\cdot 43]$$

Extending the use of the words "mediate cardinals" to cover cardinals which are comparable with Alephs up to a certain Aleph, instead of using it for non-inductive non-reflexive cardinals, we have mediate cardinals of various degrees.

$$\text{med}{}^{\backprime}\mu = \hat{\nu}[\rho < \mu\,.\supset_\rho.\,\nu > \rho\,.\sim(\nu \geq \mu\,.\,v\,.\,\nu < \mu)] \qquad \text{Df.}$$

and

$$NC\text{ med} = s{}^{\backprime}\,\text{med}{}^{\backprime\backprime}NC. \qquad \text{Df.}$$

Then

$$\text{spec}{}^{\backprime}\aleph_0 = NC\text{ ind} \cup NC\text{ refl},$$
$$\text{med}{}^{\backprime}\aleph_0 = NC - NC\text{ ind} - NC\text{ refl},$$

so that the non-inductive non-reflexive cardinals are a particular case of classes of mediate cardinals. We have

$$\text{spec}{}^{\backprime}\aleph_0 \cup \text{med}{}^{\backprime}\aleph_0 = NC, \qquad 2\cdot 01$$
$$\text{spec}{}^{\backprime}\aleph_0 \cap \text{med}{}^{\backprime}\aleph_0 = \Lambda. \qquad 2\cdot 011$$

Also

$$\text{spec}{}^{\backprime}\aleph_1 \cup (\text{med}{}^{\backprime}\aleph_0 \cup \text{med}{}^{\backprime}\aleph_1) = NC, \qquad 2\cdot 02$$
$$\text{spec}{}^{\backprime}\aleph_1 \cap (\text{med}{}^{\backprime}\aleph_0 \cup \text{med}{}^{\backprime}\aleph_1) = \Lambda, \qquad 2\cdot 021$$

and generally

$$\text{spec}{}^{\backprime}\aleph_\nu \cup (\text{med}{}^{\backprime}\aleph_0 \cup \text{med}{}^{\backprime}\aleph_1 \cdots \cup \text{med}{}^{\backprime}\aleph_\nu) = NC, \qquad 2\cdot 03$$
$$\text{spec}{}^{\backprime}\aleph_\nu \cap (\text{med}{}^{\backprime}\aleph_0 \cup \text{med}{}^{\backprime}\aleph_1 \cdots \cup \text{med}{}^{\backprime}\aleph_\nu) = \Lambda. \qquad 2\cdot 03'$$

Further

$$\text{med}{}^{\backprime}\aleph_\nu \cap \text{med}{}^{\backprime}\aleph_{\nu'} = \Lambda$$

unless $\nu = \nu'$ and $\text{spec}{}^{\backprime}\mu \subset \text{spec}{}^{\backprime}\nu$ if $\mu \geq \nu$. Hence $\text{spec}{}^{\backprime}\nu = p{}^{\backprime}\,\text{spec}{}^{\backprime\backprime}\gtrapprox{}^{\backprime\backprime}\aleph_\nu$. Therefore from $2\cdot 03\cdot 03'$ it follows that

$*2\cdot 1 \qquad \vdash.\,NC - \text{spec}{}^{\backprime}\aleph_\xi = s{}^{\backprime}\,\text{med}{}^{\backprime\backprime}\gtrapprox{}^{\backprime\backprime}\aleph_\xi.$

From 2.01.02 it follows that

$$(\text{spec}{}^{\backprime}\aleph_0 \cap \text{spec}{}^{\backprime}\aleph_1) \cup (\text{med}{}^{\backprime}\aleph_0 \cup \text{med}{}^{\backprime}\aleph_1) = NC \qquad 2\cdot 11$$

and generally from $2 \cdot 11$,

$$p' \text{ spec } `` \overrightarrow{\lesssim} ' \aleph_\xi \cup s' \text{ spec } `` \overrightarrow{\lesssim} ' \aleph_\xi = NC \qquad 2 \cdot 12$$

and

$$*2 \cdot 2 \qquad \vdash . \, NC - p' \text{ spec } `` \overrightarrow{\lesssim} ' \aleph_\xi = s' \text{ med } `` \overrightarrow{\lesssim} ' \aleph_\xi .$$

Then from $2 \cdot 1 \cdot 04$

$$\begin{aligned} s'(NC - \text{spec } `` \overrightarrow{\lesssim} ' \aleph_\xi) &= s' \text{ med } `` \overrightarrow{\lesssim} ' \aleph_\xi \\ &= s' \text{ med } `` \overrightarrow{\lesssim} ' \aleph_\xi - \text{med } ' \aleph_\xi . \end{aligned}$$

Thus

$$*2 \cdot 3 \qquad \vdash . \, s'(NC - \text{spec } `` \overrightarrow{\lesssim} `` \aleph_\xi) = NC - \text{spec } ' \aleph_\xi - \text{med } ' \aleph_\xi .$$

From $2 \cdot 1$ it follows that

$$\text{med } ' \aleph_\xi \cup \text{spec } ' \aleph_\xi \cup s' \text{ med } `` \overrightarrow{\lesssim} ' \aleph_\xi = NC.$$

Therefore from $2 \cdot 2$ we get

$$*2 \cdot 4 \qquad \vdash . \text{ med } ' \aleph_\xi \cup \text{spec } ' \aleph_\xi = p' \text{ spec } `` \overrightarrow{\lesssim} ' \aleph_\xi .$$

Taking $2 \cdot 4$ in the particular case when $\xi = \zeta + 1$, we get

$$*2 \cdot 5 \qquad \vdash . \text{ med } ' \aleph_{\zeta+1} \cup \text{spec } ' \aleph_{\zeta+1} = p' \text{ spec } `` \overrightarrow{\lesssim} ' \aleph_\zeta .$$

Further we get the proposition

$$*2 \cdot 6 \qquad \vdash : \aleph_z \in NC \text{ mult.} \quad \sim (\exists \zeta \cdot \zeta + 1 = z) \, . \supset . \text{ med } ' \aleph_z = \Lambda .$$

For suppose α is a class containing subclasses having as cardinals all Alephs less than $\aleph_z$. There will be a well-ordered series of Alephs

$$\aleph_{\eta+a_0}, \quad \aleph_{\eta+a_1}, \quad \cdots \quad \aleph_{\eta+a_\omega}, \quad \cdots$$

with $\aleph_z$ as limit. And each of the classes

$$Cl'\alpha \cap \aleph_{\eta+a_0}; \quad Cl'\alpha \cap \aleph_{\eta+a_1}; \quad \cdots \quad Cl'\alpha \cap \aleph_{\eta+a_\omega};$$

will have at least one member. The number of these classes is less than $\aleph_z$. Hence, if $\aleph_z \in NC$ mult, a selection κ can be made from this class of classes. From $1 \cdot 7$ it follows that $s'\kappa$ will be an $\aleph_z$. It is therefore clear that if a class contains subclasses having as cardinals all Alephs less than $\aleph_z$, it also contains an $\aleph_z$. There can therefore be no med $'\aleph_z$.

From $**2 \cdot 1 \cdot 6$ it follows that

$$\begin{aligned} *2 \cdot 7 \qquad \vdash : \aleph_z \in NC \text{ mult} . \sim (\exists \zeta \cdot \zeta + 1 = z) \, . \supset . \, NC - \text{spec } ' \aleph_z & \\ = s' \text{ med } `` \overrightarrow{\lesssim} ' \aleph_z &. \end{aligned}$$

$*3.$

It has been established that the following assumptions are equivalent *inter se.*

$*3 \cdot 1$ The Multiplicative Axiom.

$*3 \cdot 2$ The Comparability of Cardinals.

$*3 \cdot 3$ The Well-Orderability of Classes.*

It was also established that the multiplicative axiom implies the non-existence of med $'\aleph_0$. It was not proved and there seems no reason to suppose it is true that the non-existence of med $'\aleph_0$ implies the multiplicative axiom. The consideration of the other mediate cardinals, however, shows that the truth of the multiplicative axiom would be implied by the non-existence of med $'\aleph_0$ together with the non-existence of the mediate cardinals corresponding to Alephs other than $\aleph_0$. For we have

$$NC = \text{Aleph} \,.\, \supset \,:\, \mu \in NC \,.\, \supset \,.\, \mu \in p' \text{ spec } ``\text{Aleph}.$$

And therefore

$$NC = \text{Aleph} \,.\, \supset \,.\, NC = p' \text{ spec } ``\text{Aleph}. \qquad (3 \cdot 31)$$

Since

$$\begin{aligned} NC = \text{Aleph} \,.\, \supset \,&:\, \mu \in NC \,.\, \supset_\mu \,.\, \mu \in \text{Aleph} \\ \supset \,&:\, \mu \in NC \,.\, \supset_\mu \,.\, \exists \nu \,.\, \nu \in \text{Aleph} \,.\, \mu \in \text{spec}' \, \nu, \end{aligned}$$

therefore

$$NC = \text{Aleph} \,.\, \supset \,.\, NC = s' \text{ spec } ``NC,$$

which with $3 \cdot 31$ gives

$$NC = \text{Aleph} \,.\, \supset \,.\, NC = s' \text{ spec } ``NC = p' \text{ spec } ``NC.$$

Further

$$NC = p' \text{ spec } ``NC \,.\, \supset \,:\, \mu \in NC \cdot \nu \in NC \,.\, \supset \,.\, \mu \geq \nu \,.\, \nu \,.\, \mu < \nu.$$

This with the fact of the equivalence of $3 \cdot 2$ and $3 \cdot 3$ gives a fourth assumption which is equivalent to $3 \cdot 1$, $3 \cdot 2$ and $3 \cdot 3$, viz.,

$*3 \cdot 4$ $$NC = p' \text{ spec } ``NC = s' \text{ spec } ``NC.$$

It follows that $\mu \in NC \,.\, \supset \,.\, NC = \text{spec}' \, \mu$. Then from $2 \cdot 1$ it can be deduced that

$$\begin{aligned} Cls = 1 \cup C``\Omega \,.\, &\supset \,.\, s' \text{ med } ``\text{Aleph} = \Lambda, \\ Cls = 1 \cup Cl`\Omega \,.\, &\supset \,.\, s' \text{ med } ``NC \quad = \Lambda. \end{aligned}$$

Finally from $2 \cdot 1$, if s' med $``\text{Aleph} = \Lambda$, $\mu \in \text{Aleph} \,.\, \supset \,.\, NC = \text{spec } `\mu$. Hence $NC = p'$ spec $``\text{Aleph} = s'$ spec $``\text{Aleph}$. If $\mu \in NC - \text{Aleph}$, then there is some Aleph to whose species $\mu$ does not belong and therefore $NC \neq p'$ spec $`\text{Aleph}$. Hence if $s'$ med $``NC = \Lambda$, there are no cardinals not Alephs. Thus we have established the equivalence of the statements $3 \cdot 1$, $3 \cdot 2$ and $3 \cdot 3$ and $3 \cdot 4$ and the statement

$*3 \cdot 5$ $$NC \text{ med} = \Lambda.$$

* See "Principia Mathematica," $**258 \cdot 37 \cdot 39$, and Hartogs, "Über das Problem der Wohlordnung," *Mathematische Annalen*, Bd. LXXVI.

PERIODIC OSCILLATIONS OF THREE FINITE MASSES ABOUT THE LAGRANGIAN CIRCULAR SOLUTIONS.

By H. E. Buchanan.

Introduction.—In 1772 LaGrange announced the particular solutions of the problem of three bodies in which the ratios of the mutual distances remain constant. These are the so-called straight line and equilateral triangle solutions. Liouville, in 1845,* investigated the stability of these solutions and found that for a small displacement the bodies would, in general, depart to relatively great distances from the starting place. The question still remained whether or not these displacements could be selected so that a set of periodic orbits would result. In this paper several classes of such orbits in the vicinity of the straight line solutions are shown to exist. In case one of three bodies is infinitesimal the problem has been treated by Moulton in Chapter V of his Periodic Orbits. The method which was developed there will be applied here when all of the bodies are finite.

The Differential Equations.—The general differential equations of motion for three finite bodies whose coördinates are referred to axes rotating uniformly in the $\xi\eta$-plane are

$$\left.\begin{aligned}
\frac{d^2\xi_i}{dt^2} - 2\omega\frac{d\eta_i}{dt} &= \omega^2\xi_i - \sum_{j=1}^{3} m_j\frac{(\xi_i-\xi_j)}{r_{ij}^3} = \frac{1}{m_i}\frac{\partial U}{\partial \xi_i},\\
\frac{d^2\eta_i}{dt^2} + 2\omega\frac{d\xi_i}{dt} &= \omega^2\eta_i - \sum_{j=1}^{3} m_j\frac{(\eta_i-\eta_j)}{r_{ij}^3} = \frac{1}{m_i}\frac{\partial U}{\partial \eta_i},\\
\frac{d^2\zeta_i}{dt^2} &= -\sum_{j=1}^{3} m_j\frac{(\zeta_i-\zeta_j)}{r_{ij}^3} = \frac{1}{m_i}\frac{\partial U}{\partial \zeta_i},\\
U &= \tfrac{1}{2}\sum_{i=1}^{3}\omega^2 m_i(\xi_i^2+\eta_i^2) + \tfrac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{m_i m_j}{r_{ij}} \quad (j \neq i),
\end{aligned}\right\} \qquad (1)$$

where ω denotes the angular speed, m_i one of the masses, and where the units have been chosen so that the Gaussian constant is unity.

Let the coördinates in the straight line solutions be

$$\xi_i = \xi_i^{(0)}, \quad \eta_i = \zeta_i = 0 \quad (i = 1, 2, 3),$$

* "Connaissance des Temps," 1845.

and make the transformation

$$\xi_i = \xi_i^{(0)} + x_i', \qquad \eta_i = y_i', \qquad \zeta_i = z_i';$$

then equations (1) become

$$\left.\begin{aligned}
&\frac{d^2x_i'}{dt^2} - 2\omega\frac{dy_i'}{dt} = \omega^2(x_i' + \xi_i^{(0)}) - \sum_{j=1}^{3} m_j\frac{(x_i' + \xi_i^{(0)} - x_j' - \xi_j^{(0)})}{r_{ij}^3},\\
&\frac{d^2y_i'}{dt^2} + 2\omega\frac{dx_i'}{dt} = \omega^2 y_i' - \sum_{j=1}^{3} m_j\frac{(y_i' - y_j')}{r_{ij}^3},\\
&\qquad\frac{d^2z_i'}{dt^2} = -\sum_{j=1}^{3} m_j\frac{(z_i' - z_j')}{r_{ij}^3} \qquad (i = 1, 2, 3;\, j \neq i).
\end{aligned}\right\} \tag{2}$$

These are the equations which define the oscillations.

Expansion of the Right Members.—The right members of equations (2) will be expanded as a power series in x_i', y_i', z_i'. The region of convergence is the common region of convergence of the expansions of

$$\frac{1}{r_{12}} = \frac{1}{\sqrt{(x_2' + \xi_2^{(0)} - x_1' - \xi_1^{(0)})^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2}},$$

$$\frac{1}{r_{13}} = \frac{1}{\sqrt{(x_3' + \xi_3^{(0)} - x_1' - \xi_1^{(0)})^2 + (y_3' - y_1')^2 + (z_3' - z_1')^2}},$$

$$\frac{1}{r_{23}} = \frac{1}{\sqrt{(x_3' + \xi_3^{(0)} - x_2' - \xi_2^{(0)})^2 + (y_3' - y_2')^2 + (z_3' - z_2')^2}}.$$

The first equation can be written in the form

$$\frac{1}{r_{12}} = \frac{1}{x_2' + \xi_2^{(0)} - x_1' - \xi_1^{(0)}}\,\frac{1}{\sqrt{1 + \dfrac{(y_2' - y_1')^2 + (z_2' - z_1')^2}{(x_2' + \xi_2^{(0)} - x_1' - \xi_1^{(0)})^2}}},$$

with similar expressions for $1/r_{13}$ and $1/r_{23}$. It follows that the expansions will converge if

$$\frac{(y_2' - y_1')^2 + (z_2' - z_1')^2}{(x_2' + \xi_2^{(0)} - x_1' - \xi_1^{(0)})^2} < +1, \qquad -1 < \frac{x_2' - x_1'}{\xi_2^{(0)} - \xi_1^{(0)}} < +1,$$

$$\frac{(y_3' - y_1')^2 + (z_3' - z_1')^2}{(x_3' + \xi_3^{(0)} - x_1' + \xi_1^{(0)})^2} < +1, \qquad -1 < \frac{x_3' - x_1'}{\xi_3^{(0)} - \xi_1^{(0)}} < +1,$$

$$\frac{(y_3' - y_2')^2 + (z_3' - z_2')^2}{(x_3' + \xi_3^{(0)} - x_2' - \xi_2^{(0)})^2} < +1, \qquad -1 < \frac{x_3' - x_2'}{\xi_3^{(0)} - \xi_2^{(0)}} < +1.$$

The three conditions on the left are satisfied if the line joining any two of the bodies always makes an angle of less than 45° with the x-axis.

Inside the regions defined by these inequalities the right members can be expanded as power series in x'_i, y'_i, z'_i. Then the differential equations can be written in the form

$$\left.\begin{aligned} \frac{d^2x'_i}{dt^2} - 2\omega\frac{dy'_i}{dt} &= X_{1i} + X_{2i} + \cdots, \\ \frac{d^2y'_i}{dt^2} + 2\omega\frac{dx'_i}{dt} &= Y_{1i} + Y_{2i} + \cdots, \\ \frac{d^2z'_i}{dt^2} &= Z_{1i} + Z_{2i} + \cdots \quad (i = 1, 2, 3), \end{aligned}\right\} \tag{3}$$

where the X_{ji}, Y_{ji}, Z_{ji} are homogeneous functions of x'_i, y'_i, z'_i of degree j. Further X_{ji} is a function of y'^2_i and z'^2_i; Y_{ji} is y'_i times a function of y'^2_i and z'^2_i; Z_{ji} is z'_i times a function of y'^2_i and z'^2_i.

THE SYMMETRY THEOREM.—By making use of the properties of X_{ji}, Y_{ji} and Z_{ji} we will prove the theorem:

If all the bodies are projected from and at right angles to the x-axis, the orbits will be symmetrical with respect to this axis geometrically and in the time.

If the initial projection is from and at right angles to the x-axis, then

$$x'_i(0) = a_i, \qquad \frac{dx'_i}{dt}(0) = 0, \qquad y'_i(0) = 0, \qquad \frac{dy'_i}{dt}(0) = b_i, \qquad z'_i(0) = 0,$$
$$\frac{dz'_i}{dt}(0) = c_i,$$

and the solutions may be written

$$\begin{aligned} x'_i &= \varphi_i(a_i, b_i, c_i, t), \\ y'_i &= \psi(a_i, b_i, c_i, t), \\ z'_i &= \theta(a_i, b_i, c_i, t) \quad (i = 1, 2, 3). \end{aligned}$$

Equations (3) remain unchanged when t is replaced by $-t$; y'_i by $-y'_i$ and z'_i by $-z'_i$; but the solutions for the same initial conditions become

$$\begin{aligned} x'_i &= \varphi_i(a_i, b_i, c_i, -t), \\ y'_i &= -\psi_i(a_i, b_i, c_i, -t), \\ z'_i &= -\theta_i(a_i, b_i, c_i, -t). \end{aligned}$$

The φ_i are therefore even functions of t while the ψ_i and the θ_i are odd functions of t.

The Parameters ϵ and δ.—Applying to equations (3) the transformation

$$x'_i = \epsilon' x_i, \qquad y'_i = \epsilon' y_i, \qquad z'_i = \epsilon' z_i, \qquad t = (1 + \delta)\tau,$$

we get

$$\left.\begin{aligned} \frac{d^2x_i}{d\tau^2} - 2\omega(1+\delta)\frac{dy_i}{d\tau} &= (1+\delta)^2\{X_{1i} + \epsilon' X_{2i} + \cdots\}, \\ \frac{d^2y_i}{d\tau^2} + 2\omega(1+\delta)\frac{dx_i}{d\tau} &= (1+\delta)^2\{Y_{1i} + \epsilon' Y_{2i} + \cdots\}, \\ \frac{d^2z_i}{d\tau^2} &= (1+\delta)^2\{Z_{1i} + \epsilon' Z_{2i} + \cdots\} \quad (i = 1, 2, 3). \end{aligned}\right\} \tag{4}$$

These equations are valid for the physical problem proposed so long as the bodies remain in the regions of convergence previously determined. Let us generalize the problem by replacing ϵ' by ϵ, a parameter which can take all values in the neighborhood of zero. We get a solution of the physical problem only when $\epsilon' = \epsilon$. The method will be to find all periodic solutions when $\epsilon = \delta = 0$ and discuss the analytic continuation of these when ϵ increases to the value ϵ'.

The Center of Gravity Integrals and the Characteristic Exponents.— Equations (4) possess six integrals which define the position and motion of the center of gravity. When the origin is at the center of gravity, as it is in equations (4), these integrals are

$$\begin{gathered} m_1x_1 + m_2x_2 + m_3x_3 = 0, \qquad m_1y_1 + m_2y_2 + m_3y_3 = 0, \\ m_1z_1 + m_2z_2 + m_3z_3 = 0 \end{gathered} \tag{5}$$

and their first derivatives. By means of these equations it is possible to eliminate three of the nine equations of (4). We choose to eliminate the x_2, y_2 and z_2 equations, so that hereafter i takes only the values 1 and 3.

To obtain the generating solutions we put $\epsilon = \delta = 0$ in equations (4). Evidently all except the first terms of the right members disappear. If we remember that x_2, y_2 and z_2 have been eliminated by (5), we obtain the linear homogeneous differential equations

$$\left.\begin{aligned} \frac{d^2x_1}{d\tau^2} - 2\omega\frac{dy_1}{d\tau} &= (\omega^2 + 2A_1)x_1 + 2B_1x_3, \\ \frac{d^2x_3}{d\tau^2} - 2\omega\frac{dy_3}{d\tau} &= 2B_3x_1 + (\omega^2 + 2A_3)x_3, \\ \frac{d^2y_1}{d\tau^2} + 2\omega\frac{dx_1}{d\tau} &= (\omega^2 - A_1)y_1 - B_1y_3, \\ \frac{d^2y_3}{d\tau^2} + 2\omega\frac{dx_3}{d\tau} &= -B_3y_1 + (\omega^2 - A_3)y_3, \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned}\frac{d^2z_1}{d\tau^2} &= -A_1z_1 - B_1z_3,\\ \frac{d^2z_3}{d\tau^2} &= -B_3z_1 - A_3z_3,\end{aligned}\right\} \tag{7}$$

where

$$A_1 = \frac{m_1 + m_2}{r_{12}^3} + \frac{m_3}{r_{13}^3}, \qquad A_3 = \frac{m_1}{r_{13}^3} + \frac{m_2 + m_3}{r_{23}^3},$$

$$B_1 = m_3\left(\frac{1}{r_{12}^3} - \frac{1}{r_{13}^3}\right), \qquad B_3 = m_1\left(\frac{1}{r_{23}^3} - \frac{1}{r_{13}^3}\right),$$

and the r_{ij} belong to the circular solutions.

Equations (6) and (7) are mutually independent; therefore we can treat them separately. Let us substitute

$$x_i = K_ie^{\lambda\tau}, \qquad y_i = L_ie^{\lambda\tau}, \qquad z_i = M_ie^{\sigma\tau}.$$

From equations (6) there results a set of equations linear and homogeneous in K_i and L_i, and from equations (7) a set linear and homogeneous in M_i. In order that there shall exist a solution other than that given by $K_i = L_i = M_i = 0$ we must have

$$\begin{vmatrix} \lambda^2 - \omega^2 - 2A_1, & -2B_1, & -2\lambda\omega, & 0 \\ -2B_3, & \lambda^2 - \omega^2 - 2A_3, & 0, & -2\lambda\omega \\ 2\lambda\omega, & 0, & \lambda^2 - \omega^2 + A_1, & B_1 \\ 0, & 2\lambda\omega, & B_3, & \lambda^2 - \omega^2 + A_3 \end{vmatrix} = 0, \tag{8}$$

and

$$\begin{vmatrix} \sigma^2 + A_1, & B_1 \\ B_3, & \sigma^2 + A_3 \end{vmatrix} = 0. \tag{9}$$

The left member of equation (8) is a function of λ^2 for if λ be changed into $-\lambda$ the function is unchanged. Lagrange* has proved that there are elliptical orbits in which the bodies all remain on a straight line. These ought to appear here as oscillations near the circular solutions. Therefore we expect equations (8) to have a solution $\lambda^2 = -\omega^2$. On substituting $-\omega^2$ for λ^2 in (8) we obtain from the left member

$$\begin{vmatrix} A_1, & B_1 \\ B_3, & A_3 \end{vmatrix} \times \begin{vmatrix} \omega^2 - A_1, & B_1 \\ B_3, & \omega^2 - A_3 \end{vmatrix}. \tag{10}$$

The quantities ω^2, A_1, A_3, B_1 and B_3 are not independent, but are related

* Tisserand, "Méchanique Céleste," Vol. 1, Chapter 8; Moulton's "Celestial Mechanics," p. 217.

by the equations which determine the circular solution, namely,

$$\omega^2\xi_1^{(0)} - \frac{\xi_1^{(0)}(m_1 + m_2) + m_3\xi_3^{(0)}}{r_{12}^3} - \frac{m_3(\xi_1^{(0)} - \xi_3^{(0)})}{r_{13}^3} = 0,$$

$$\omega^2\xi_3^{(0)} - \frac{m_1(\xi_3^{(0)} - \xi_1^{(0)})}{r_{13}^3} - \frac{\xi_3^{(0)}(m_2 + m_3) + m_1\xi_1^{(0)}}{r_{23}^3} = 0,$$

which become when expressed in terms of A's and B's

$$\omega^2 - A_1 = \frac{\xi_3^{(0)}}{\xi_1^{(0)}} B_1, \qquad \omega^2 - A_3 = \frac{\xi_1^{(0)}}{\xi_3^{(0)}} B_3. \tag{11}$$

The first part of (10) is clearly positive. If equations (11) are used the second factor vanishes. Therefore $-\omega^2$ is a root of (8). Further, if we put $\lambda^2 = 0$ in (8) the determinant breaks up into the product of two determinants one of which is the second factor of (10). Therefore (8) has a root $\lambda^2 = 0$ and can be reduced to a quadratic in λ^2 the exact form of which is

$$\lambda^4 + \lambda^2(3\omega^2 - A_1 - A_3) + 3\omega^4 + A_1A_3 - 2A_1^2 - 2A_3^2 - 5B_1B_3 = 0. \tag{12}$$

Since B_1 and B_3 are positive and $\xi_1^{(0)}$ has the opposite sign to $\xi_3^{(0)}$ equations (11) show that

$$\omega^2 - A_1 < 0, \qquad \omega^2 - A_3 < 0. \tag{13}$$

Equations (11) also give

$$(\omega^2 - A_1)(\omega^2 - A_3) = B_1B_3. \tag{14}$$

Using the relation (14) the constant term of (12) can be grouped

$$(\omega^2 + A_3)(\omega^2 - A_1) + (\omega^2 + A_1)(\omega^2 - A_3) + A_1(\omega^2 - A_1) \\ + A_3(\omega^2 - A_3) - 4B_1B_3.$$

The constant term of (12) is therefore negative and the solutions for λ^2 are real, one positive and one negative. Call these roots $-\rho_1^2$ and ρ_2^2.

The forms of equations (9) and (10) show that (9) also has the root $\sigma^2 = -\omega^2$. The other root of (9) is $\sigma^2 = \omega^2 - A_1 - A_3$ which is negative. We denote $\omega^2 - A_1 - A_3$ by $-\nu^2$. From these specific values of the roots of (9) it is clear that $-\nu^2$ is less than $-\omega^2$. If λ^2 in (12) is replaced by $-\nu^2$, the left member reduces to $\omega^2(2\omega^2 - A_1 - A_3)$ which is negative. Hence $-\rho_1^2 < -\nu^2$ and the relative magnitude of the roots of (8) and (9) whatever the values of m_1, m_2 and m_3 are as follows:

$$-\rho_1^2 < -\nu^2 < -\omega^2 < 0 < \rho_2^2.$$

The Generating Solution.—Having found the characteristic exponents,

we can write the general solutions of equations (6) and (7). They are

$$\left.\begin{aligned}
x_1&=K_{11}e^{\rho_1\iota\tau}+K_{12}e^{-\rho_1\iota\tau}+K_{13}e^{\omega\iota\tau}+K_{14}e^{-\omega\iota\tau}+K_{15}e^{\rho_2\tau}+K_{16}e^{-\rho_2\tau}+K_{17}+K_{18}\tau,\\
x_3&=K_{21}e^{\rho_1\iota\tau}+K_{22}e^{-\rho_1\iota\tau}+K_{23}e^{\omega\iota\tau}+K_{24}e^{-\omega\iota\tau}+K_{25}e^{\rho_2\tau}+K_{26}e^{-\rho_2\tau}+K_{27}+K_{28}\tau,\\
y_1&=L_{11}e^{\rho_1\iota\tau}+L_{12}e^{-\rho_1\iota\tau}+L_{13}e^{\omega\iota\tau}+L_{14}e^{-\omega\iota\tau}+L_{15}e^{\rho_2\tau}+L_{16}e^{-\rho_2\tau}+L_{17}+L_{18}\tau,\\
y_3&=L_{21}e^{\rho_1\iota\tau}+L_{22}e^{-\rho_1\iota\tau}+L_{23}e^{\omega\iota\tau}+L_{24}e^{-\omega\iota\tau}+L_{25}e^{\rho_2\tau}+L_{26}e^{-\rho_2\tau}+L_{27}+L_{28}\tau,\\
z_1&=M_{11}e^{\omega\iota\tau}+M_{12}e^{-\omega\iota\tau}+M_{13}e^{\nu\iota\tau}+M_{14}e^{-\nu\iota\tau},\\
z_3&=M_{21}e^{\omega\iota\tau}+M_{22}e^{-\omega\iota\tau}+M_{23}e^{\nu\iota\tau}+M_{24}e^{-\nu\iota\tau}.
\end{aligned}\right\}(15)$$

The solutions for x_i and y_i can contain only eight arbitrary constants. We may choose for six of them $L_{11}\cdots L_{16}$. Then the L_{3j}, K_{1j} and K_{3j}, $j=1\cdots 6$, are uniquely determined in terms of $L_{11}\cdots L_{16}$ by any three of the equations

$$\left.\begin{aligned}
(\lambda^2-\omega^2-2A_1)K_{1j}-2B_1K_{3j}-2\lambda\omega L_{1j}&=0,\\
-2B_3K_{1j}+(\lambda^2-\omega^2-2A_3)K_{3j}-2\lambda\omega L_{3j}&=0,\\
2\lambda\omega K_{1j}+(\lambda^2-\omega^2+A_1)L_{1j}+B_1L_{3j}&=0,\\
2\lambda\omega K_{3j}+B_3L_{1j}+(\lambda^2-\omega^2+A_3)L_{3j}&=0\quad(j=1\cdots 6),
\end{aligned}\right\}(16)$$

because $\rho_1\iota$, $-\rho_1\iota$, $\omega\iota$, $-\omega\iota$, ρ_2, and $-\rho_2$ are simple roots of (8) and therefore not all the first minors obtained by suppressing the column belonging to the L_{ij} can vanish.

To find K_{i7}, K_{i8}, L_{i7} and L_{i8} substitute $x_i=K_{i7}+K_{i8}\tau$, $y_i=L_{i7}+L_{i8}\tau$ in the differential equations (6). There results

$$\begin{aligned}
-2\omega L_{18}&=(\omega^2+2A_1)(K_{17}+K_{18}\tau)+2B_1(K_{37}+K_{38}\tau),\\
-2\omega L_{38}&=2B_3(K_{17}+K_{18}\tau)+(\omega^2+2A_3)(K_{37}+K_{38}\tau),\\
2\omega K_{18}&=(\omega^2-A_1)(L_{17}+L_{18}\tau)-B_1(L_{37}+L_{38}\tau),\\
2\omega K_{38}&=-B_3(L_{17}+L_{18}\tau)+(\omega^2-A_3)(L_{37}+L_{38}\tau).
\end{aligned}$$

Since these equations are identities in τ we have

$$\left.\begin{aligned}
2\omega L_{18}+(\omega^2+2A_1)K_{17}+2B_1K_{37}&=0,\\
2\omega L_{38}+2B_3K_{17}+(\omega^2+2A_3)K_{37}&=0,
\end{aligned}\right\}(17)$$

$$\left.\begin{aligned}
2\omega K_{18}-(\omega^2-A_1)L_{17}+B_1L_{37}&=0,\\
2\omega K_{38}+B_3L_{17}-(\omega^2-A_3)L_{37}&=0,
\end{aligned}\right\}(18)$$

$$\left.\begin{aligned}
(\omega^2+2A_1)K_{18}+2B_1K_{38}&=0,\\
2B_3K_{18}+(\omega^2+2A_3)K_{38}&=0,
\end{aligned}\right\}(19)$$

$$\left.\begin{aligned}
(\omega^2-A_1)L_{18}-B_1L_{38}&=0,\\
-B_3L_{18}+(\omega^2-A_3)L_{38}&=0.
\end{aligned}\right\}(20)$$

In (19) the determinant of the K_{i8} is

$$\begin{vmatrix} \omega^2 + 2A_1, & 2B_1 \\ 2B_3, & \omega^2 + 2A_3 \end{vmatrix} \equiv \omega^4 + 2\omega^2(A_1 + A_3) + 4(A_1A_3 - B_1B_3),$$

which is positive. Therefore $K_{18} = K_{38} = 0$. In (20) the determinant of L_{i8} vanishes, therefore

$$L_{38} = \frac{\omega^2 - A_1}{B_1} L_{18} = \frac{B_3}{\omega^2 - A_3} L_{18} = \frac{\xi_3^{(0)}}{\xi_1^{(0)}} L_{18}.$$

Since $K_{18} = K_{38} = 0$ it follows from (18) that

$$L_{37} = \frac{\xi_3^{(0)}}{\xi_1^{(0)}} L_{17}.$$

Then (17) combined with (14) gives

$$K_{17} = -\frac{2}{3\omega} L_{18}, \qquad K_{37} = -\frac{2}{3\omega}\frac{\xi_3^{(0)}}{\xi_1^{(0)}} L_{18}.$$

Hence we choose L_{17} and L_{18} arbitrarily. In the same way we choose M_{1j} $(j = 1 \cdots 4)$ arbitrarily and determine M_{3j} in terms of M_{1j} by either of the equations

$$(\sigma^2 + A_1)M_{1j} + B_1 M_{3j} = 0, \qquad B_3 M_{1j} + (\sigma^2 + A_3)M_{3j} = 0,$$

which give

$$M_{31} = \frac{\xi_3^{(0)}}{\xi_1^{(0)}} M_{11}, \qquad M_{32} = \frac{\xi_3^{(0)}}{\xi_1^{(0)}} M_{12}, \qquad M_{33} = \frac{\xi_1^{(0)}}{\xi_3^{(0)}}\frac{B_3}{B_1} M_{13},$$
$$M_{34} = \frac{\xi_1^{(0)}}{\xi_3^{(0)}}\frac{B_3}{B_1} M_{14}. \tag{21}$$

When $dx_i/d\tau = y_i = z_i = 0$ $(i = 1, 3)$ at $\tau = 0$, we get for the solutions

$$x_i = K_{i1} \cos \rho_1\tau + K_{i3} \cos \omega\tau + K_{i5} \cosh \rho_2\tau + K_{i7},$$
$$y_i = L_{i1} \sin \rho_1\tau + L_{i3} \sin \omega\tau + L_{i5} \sinh \rho_2\tau + L_{i8}\tau,$$
$$z_i = M_{i1} \sin \omega\tau + M_{i3} \sin \nu\tau.$$

The generating solutions must be periodic. Therefore we have the following cases to consider:

(*a*) ρ_1, ω and ν incommensurable.

Case 1. $x_i = y_i = 0, \qquad z_i = M_{i3} \sin \nu\tau.$

Case 2. $x_i = K_{i3} \cos \omega\tau, \qquad y_i = L_{i3} \sin \omega\tau, \qquad z_i = M_{i1} \sin \omega\tau.$

Case 3. $x_i = K_{i1} \cos \rho_1\tau, \qquad y_i = L_{i1} \sin \rho_1\tau, \qquad z_i = 0.$

(*b*) ω and ν commensurable but ρ_1 incommensurable with them.

Case 4. $x_i = K_{i3} \cos \omega\tau, \quad y_i = L_{i3} \sin \omega\tau, \quad z_i = M_{i1} \sin \omega\tau + M_{i3} \sin \nu\tau.$

(*c*) ρ_1 and ν commensurable, but ω incommensurable with them.

Case 5. $x_i = K_{i1} \cos \rho_1\tau, \qquad y_i = L_{i1} \sin \rho_1\tau, \qquad z_i = M_{i3} \sin \nu\tau.$

(*d*) ρ_1 and ω commensurable, but ν incommensurable with them.

Case 6. $x_i = K_{i1} \cos \rho_1\tau + K_{i3} \cos \omega\tau, \quad y_i = L_{i1} \sin \rho_1\tau + L_{i3} \sin \omega\tau,$
$z_i = M_{i1} \sin \omega\tau.$

(*e*) ρ_1, ω and ν commensurable.

Case 7. $x_i = K_{i1} \cos \rho_1\tau + K_{i3} \cos \omega\tau, \quad y_i = L_{i1} \sin \rho_1\tau + L_{i3} \sin \omega\tau,$
$z_i = M_{i1} \sin \omega\tau + M_{i3} \sin \nu\tau.$

Transformation to the Normal Form.—For the existence proofs and for certain parts of the constructions of the solutions the variables of equations (4) are inconvenient. To introduce variables better adapted to our purpose we make a transformation of the form

$$x_1 = \sum_{j=1}^{8} a_{1j}u_j, \qquad x_1' = \sum_{j=1}^{8} a_{2j}u_j, \qquad x_3 = \sum_{j=1}^{8} a_{3j}u_j, \qquad \text{etc.,}$$

where the primes denote derivatives with respect to τ. The determinant of the substitution must not be zero. It follows from the general theory of differential equations in which the characteristic equation has a double root zero that it is possible to determine the a_{ij} so that equations (4) assume the form

$$\left.\begin{aligned}
u_1' &= +\,(1+\delta)\rho_1\iota u_1 + (1+\delta)\epsilon P_1(u_j, z_i),\\
u_2' &= -\,(1+\delta)\rho_1\iota u_2 + (1+\delta)\epsilon P_2(u_j, z_i),\\
u_3' &= +\,(1+\delta)\omega\iota u_3 + (1+\delta)\epsilon P_3(u_j, z_i),\\
u_4' &= -\,(1+\delta)\omega\iota u_4 + (1+\delta)\epsilon P_4(u_j, z_i),\\
u_5' &= +\,(1+\delta)\rho_2 u_5 + (1+\delta)\epsilon P_5(u_j, z_i),\\
u_6' &= -\,(1+\delta)\rho_2 u_6 + (1+\delta)\epsilon P_6(u_j, z_i),\\
u_7' &= +\,(1+\delta)u_8 + (1+\delta)\epsilon P_7(u_j, z_i),\\
u_8' &= + (1+\delta)\epsilon P_8(u_j, z_i) \quad (j = 1 \cdots 8),\ (i = 1, 3),\\
z_i'' &= (1+\delta)^2\{z_{1i} + \epsilon z_{2i} + \cdots\} \quad (i = 1, 3),
\end{aligned}\right\} \quad (22)$$

where the P_j are power series in the u_j beginning with terms of the second degree. To determine the constants a_{ij} integrate equations (22) after all terms have been dropped from the right members except the linear ones.

The result is

$$u_1 = L_{11}e^{(1+\delta)\rho_1\iota\tau}, \qquad u_2 = L_{12}e^{-(1+\delta)\rho_1\iota\tau}, \qquad u_3 = L_{13}e^{(1+\delta)\omega\iota\tau},$$
$$u_4 = L_{14}e^{-(1+\delta)\omega\iota\tau}, \qquad u_5 = L_{15}e^{(1+\delta)\rho_2\tau}, \qquad u_6 = L_{16}e^{-(1+\delta)\rho_2\tau},$$
$$u_7 = L_{17} + L_{18}(1+\delta)\tau, \qquad u_8 = L_{18}.$$

On adding these eight equations we get the value of y_1 obtained from integrating the linear terms of (4). The values of x_1, x_1', y_1', y_3 and y_3' are found by multiplying the L_{1j} by certain constants obtained from solving equations (16) for K_{ij} and L_{3j} in terms of L_{1j}. The L_{1j} are arbitrary constants of integration; therefore we choose them all equal to unity since this choice obviously does not make the determinant of the substitution zero. This gives the following form for the substitution:

$$\left.\begin{aligned}
x_1 &= a_1(u_1 - u_2) + a_3(u_3 - u_4) + a_5(u_5 - u_6) + a_7u_8,\\
x_1' &= \rho_1\iota a_1(u_1 + u_2) + \omega\iota a_3(u_3 + u_4) + \rho_2 a_5(u_5 + u_6),\\
x_3 &= b_1(u_1 - u_2) + b_3(u_3 - u_4) + b_5(u_5 - u_6) + b_7u_8,\\
x_3' &= \rho_1\iota b_1(u_1 + u_2) + \omega\iota b_3(u_3 + u_4) + \rho_2 b_5(u_5 + u_6),\\
y_1 &= u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 + u_8,\\
y_1' &= \rho_1\iota(u_1 - u_2) + \omega\iota(u_3 - u_4) + \rho_2(u_5 - u_6) + u_8,\\
y_3 &= c_1(u_1 + u_2) + c_3(u_3 + u_4) + c_5(u_5 + u_6) + c_7(u_7 + u_8),\\
y_3' &= \rho_1\iota c_1(u_1 - u_2) + \omega\iota c_3(u_3 - u_4) + \rho_2 c_5(u_5 - u_6) + c_7u_8,
\end{aligned}\right\} \tag{23}$$

where the a_j, b_j and c_j $(j = 1, 3, 5)$ are the ratios of the K_{1j}, K_{3j} and L_{3j} to L_{1j}, a_7 and b_7 are the ratios of K_{17} and K_{37} to L_{18}, and c_7 is the ratio of L_{37} to L_{17}.

Equations (22) are the normal forms for the equations of motion. In the normal variables the generating solutions become

(1) $u_j = 0 \quad (j = 1 \cdots 8), \qquad z_i = M_i \sin \nu\tau \quad (i = 1, 3),$

(2) $u_j = 0 \quad (j = 1, 2, 5, 6, 7, 8), \qquad u_3 = ae^{\omega\iota\tau}, \qquad u_4 = ae^{-\omega\iota\tau},$
$z_i = M_i \sin \omega\tau \quad (i = 1, 3),$

(3) $u_1 = ae^{\rho_1\iota\tau}, \qquad u_2 = ae^{-\rho_1\iota\tau}, \qquad u_j = 0 \quad (j = 3, 4, 5, 6, 7, 8),$
$z_i = 0, \quad (i = 1, 3),$

(4) $u_j = 0 \quad (j = 1, 2, 5, 6, 7, 8), \qquad u_3 = ae^{\omega\iota\tau}, \qquad u_4 = ae^{-\omega\iota\tau},$
$z_i = M_i \sin \omega\tau + N_i \sin \nu\tau \quad (i = 1, 3),$

(5) $u_1 = ae^{\rho_1\iota\tau}, \qquad u_2 = ae^{-\rho_1\iota\tau}, \qquad u_j = 0 \quad (j = 3, 4, 5, 6, 7, 8),$
$z_i = M_i \sin \nu\tau \quad (i = 1, 3),$

(6) $u_1 = a_1e^{\rho_1\iota\tau}, \qquad u_2 = a_1e^{-\rho_1\iota\tau}, \qquad u_3 = a_2e^{\omega\iota\tau}, \qquad u_4 = a_2e^{-\omega\iota\tau},$
$u_j = 0 \quad (j = 5, 6, 7, 8), \qquad z_i = M_i \sin \nu\tau,$

(7) $u_1 = a_1e^{\rho_1\iota\tau}, \qquad u_2 = a_1e^{-\rho_1\iota\tau}, \qquad u_3 = a_2e^{\omega\iota\tau}, \qquad u_4 = a_2e^{-\omega\iota\tau},$
$u_j = 0 \quad (j = 5, 6, 7, 8), \qquad z_i = M_i \sin \omega\tau + N_i \sin \nu\tau \quad (i = 1, 3).$

The Periodicity Equations for Solutions with Period $2\pi/\nu$.—According to known theorems* on differential equations it is possible to integrate equations (22) as power series ϵ, δ, α_i, β_i and γ_i, where the α_i are the initial values of the u_i; β_i of z_i and $\nu(M_i + \gamma_i)$ of the z_i'. Instead of integrating as a multiple power series in all these parameters, it is more convenient in the computation to develop the solutions in the form

$$u_i = \sum_{j=0}^{\infty} u_{ij}\epsilon^j, \qquad z_i = \sum_{j=0}^{\infty} z_{ij}\epsilon^j, \tag{24}$$

introducing δ in connection with τ, and the α_i, β_i and γ_i by means of the initial conditions.

The equations determining u_{i0} and z_{i0} are

$$\left.\begin{aligned} u_{10}' &= +\,(1+\delta)\rho_1\iota u_{10}, & u_{50}' &= +\,(1+\delta)\rho_2 u_{50},\\ u_{20}' &= -\,(1+\delta)\rho_1\iota u_{20}, & u_{60}' &= -\,(1+\delta)\rho_2 u_{60},\\ u_{30}' &= +\,(1+\delta)\omega\iota u_{30}, & u_{70}' &= +\,(1+\delta)u_{80},\\ u_{40}' &= -\,(1+\delta)\omega\iota u_{40}, & u_{80}' &= 0,\\ z_{10}'' &= -\,A_1 z_{10} - B_1 z_{30}, & z_{30}'' &= -\,B_3 z_{10} - A_3 z_{30}. \end{aligned}\right\} \tag{25}$$

Integrating and imposing the initial conditions we have

$$\begin{aligned} u_{10} &= \alpha_1 e^{+(1+\delta)\rho_1\iota\tau}, & u_{50} &= \alpha_5 e^{+(1+\delta)\rho_2\tau},\\ u_{20} &= \alpha_2 e^{-(1+\delta)\rho_1\iota\tau}, & u_{60} &= \alpha_6 e^{-(1+\delta)\rho_2\tau},\\ u_{30} &+ \alpha_3 e^{+(1+\delta)\omega\iota\tau}, & u_{70} &= \alpha_7 + \alpha_8(1+\delta)\tau,\\ u_{40} &= \alpha_4 e^{-(1+\delta)\omega\iota\tau}, & u_{80} &= \alpha_8, \end{aligned}$$

$$z_{i0} = M_{i1}e^{(1+\delta)\omega\iota\tau} + M_{i2}e^{-(1+\delta)\omega\iota\tau} + M_{i3}e^{(1+\delta)\nu\iota\tau} + M_{i4}e^{-(1+\delta)\nu\iota\tau},$$

where by equations (21) the M_{3j} are expressed in terms of the M_{1j}. The M_{1j} are expressed in terms of the initial conditions by the equations

$$\beta_1 = M_{11} + M_{12} + M_{13} + M_{14},$$

$$\beta_3 = (M_{11} + M_{12})\frac{\omega^2 - A_1}{B_1} + (M_{13} + M_{14})\frac{\nu^2 - A_1}{B_1},$$

$$\frac{\nu(M_1 + \gamma_1)}{(1+\delta)\iota} = (M_{11} - M_{12})\omega + (M_{13} - M_{14})\nu,$$

$$\frac{\nu(M_3 + \gamma_3)}{(1+\delta)\iota} = (M_{11} - M_{12})\omega\frac{\omega^2 - A_1}{B_1} + (M_{13} - M_{14})\nu\frac{\nu^2 - A_1}{B_1}.$$

The solutions of (22) are

* Moulton's "Periodic Orbits," par. 9.

$$
\left.\begin{aligned}
u_1 &= \alpha_1 e^{(1+\delta)\rho_1 \iota\tau} + \epsilon P_1(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_2 &= \alpha_2 e^{-(1+\delta)\rho_1 \iota\tau} + \epsilon P_2(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_3 &= \alpha_3 e^{(1+\delta)\omega\iota\tau} + \epsilon P_3(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_4 &= \alpha_4 e^{-(1+\delta)\omega\iota\tau} + \epsilon P_4(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_5 &= \alpha_5 e^{(1+\delta)\rho_2\tau} + \epsilon P_5(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_6 &= \alpha_6 e^{-(1+\delta)\rho_2\tau} + \epsilon P_6(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_7 &= \alpha_7 + \alpha_8(1+\delta)\tau + \epsilon P_7(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_8 &= \alpha_8 + \epsilon P_8(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
z_i &= M_{i1} e^{(1+\delta)\omega\iota\tau} + M_{i2} e^{-(1+\delta)\omega\iota\tau} + M_{i3} e^{(1+\delta)\nu\iota\tau} + M_{i4} e^{-(1+\delta)\nu\iota\tau}\\
&\qquad\qquad + \epsilon Q_i(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
z_i' &= (1+\delta)\iota[(M_{i1} e^{+(1+\delta)\omega\iota\tau} - M_{i2} e^{-(1+\delta)\omega\iota\tau})\omega\\
&\qquad + (M_{i3} e^{(1+\delta)\nu\iota\tau} - M_{i4} e^{-(1+\delta)\nu\iota\tau})\nu] + \epsilon Q_i'(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),
\end{aligned}\right\} \quad (28)
$$

where P_i, Q_i and Q_i' are power series in the indicated arguments.

Sufficient conditions for a periodic solution having the period $2\pi/\nu$ are

$$
\left.\begin{aligned}
u_i\left(\frac{2\pi}{\nu}\right) - u_i(0) = 0 \quad (i = 1 \cdots 8),\\
z_i\left(\frac{2\pi}{\nu}\right) - z_i(0) = 0, \qquad z_i'\left(\frac{2\pi}{\nu}\right) - z_i'(0) = 0 \quad (i = 1, 3).
\end{aligned}\right\} \quad (29)
$$

The Existence of Solutions with Period $2\pi/\nu$.—Of the ten known integrals, we have already made use of the six connected with the center of gravity. There are four more to be considered, the energy integral and the three integrals of areas. By means of these four integrals it will be shown that four of the twelve equations of (29) are redundant and can be suppressed.

The integrals are explicitly

$$
\left.\begin{aligned}
F_1 &= \sum_{i=1}^{3} m_i(\bar{x}_i\bar{y}_i' - \bar{y}_i\bar{x}_i') - c_1 = 0,\\
F_2 &= \sum_{i=1}^{3} m_i(\bar{y}_i\bar{z}_i' - \bar{z}_i\bar{y}_i') - c_2 = 0,\\
F_3 &= \sum_{i=1}^{3} m_i(\bar{z}_i\bar{x}_i' - \bar{x}_i\bar{z}_i') - c_3 = 0,\\
F_4 &= \tfrac{1}{2}\sum_{i=1}^{3} [(\bar{x}_i')^2 + (\bar{y}_i')^2 + (\bar{z}_i')^2],
\end{aligned}\right\} \quad (30)
$$

where the $\bar{x}_i$, $\bar{y}_i$ and $\bar{z}_i$ are the rectangular coördinates referred to fixed axes whose origin is at the center of gravity of the system. These variables are related to those used in (22) through the equations

$$
\left.\begin{aligned}
\bar{x}_i &= \xi_i \cos \omega t - \eta_i \sin \omega t,\\
\bar{y}_i &= \xi_i \sin \omega t + \eta_i \cos \omega t,\\
\xi_i &= \xi_i^{(0)} + \epsilon x_i, \qquad \eta_i = \epsilon y_i,\\
\bar{z}_i &= \epsilon z_i, \qquad t = (1+\delta)\tau, \text{ equations (23).}
\end{aligned}\right\} \quad (31)
$$

Now let $u_i = \alpha_i + v_i$ $(i = 1 \cdots 8)$,

$$z_1 = (M_1 + \gamma_1)\sin\nu\tau + \zeta_1, \qquad z_3 = (M_3 + \gamma_3)\sin\nu\tau + \zeta_3,$$
$$z_1' = \nu(M_1 + \gamma_1)\cos\nu\tau + \zeta_1', \qquad z_3' = \nu(M_3 + \gamma_3)\cos\nu\tau + \zeta_3',$$

where, from (21), $M_3 = [(\nu^2 - A_1)/B_1]M_1$ and where M_1 is taken distinct from zero. It follows from the initial conditions adopted at the beginning of this article that

$$v_1(0) = v_2(0) = \cdots v_8(0) = \zeta_1(0) = \zeta_3(0) = \zeta_1'(0) = \zeta_3'(0) = 0.$$

Since c_1, c_2, c_3 and h can be expressed in terms of the α_i, β_i and γ_i as power series, the integrals can be written in the form

$$F_i = F_i(v_1 \cdots v_8, \zeta_1, \zeta_3, \zeta_1', \zeta_3', \alpha_1 \cdots \alpha_8, \beta_1, \beta_3, \gamma_1, \gamma_3) = 0 \quad (i = 1 \cdots 4),$$

where ϵ is divided out when it enters as a factor. These equations are satisfied at $\tau = 2\pi/\nu$ by $v_1 = \cdots = v_8 = \zeta_1 = \zeta_3 = \zeta_1' = \zeta_3' = 0$ whatever may be the values of $\alpha_1, \cdots \alpha_8, \beta_1, \beta_3, \gamma_1$ and γ_3. For this value of τ they can be solved for v_8, ζ_3, ζ_1', and ζ_3' as power series in $v_1 \cdots v_7, \zeta_1, \alpha_1 \cdots \alpha_8$, $\beta_1, \beta_3, \gamma_1$ and γ_3, and the solutions will vanish for $v_1 = \cdots = v_7 = \zeta_1 = 0$ whatever $\alpha_1 \cdots \alpha_1, \beta_1, \beta_3, \gamma_1$ and γ_3 may be, provided the determinant

$$D = \begin{vmatrix} \dfrac{\partial F_1}{\partial v_8}, & \dfrac{\partial F_1}{\partial \zeta_3}, & \dfrac{\partial F_1}{\partial \zeta_1'}, & \dfrac{\partial F_1}{\partial \zeta_3'} \\ \dfrac{\partial F_2}{\partial v_8}, & \dfrac{\partial F_2}{\partial \zeta_3}, & \dfrac{\partial F_2}{\partial \zeta_1'}, & \dfrac{\partial F_2}{\partial \zeta_3'} \\ \dfrac{\partial F_3}{\partial v_8}, & \dfrac{\partial F_3}{\partial \zeta_3}, & \dfrac{\partial F_3}{\partial \zeta_1'}, & \dfrac{\partial F_3}{\partial \zeta_3'} \\ \dfrac{\partial F_4}{\partial v_8}, & \dfrac{\partial F_4}{\partial \zeta_3}, & \dfrac{\partial F_4}{\partial \zeta_1'}, & \dfrac{\partial F_4}{\partial \zeta_3'} \end{vmatrix} \tag{32}$$

is distinct from zero for $v_1 = \cdots = v_8 = \zeta_1 = \zeta_3 = \zeta_1' = \zeta_3' = \alpha_1 \cdots = \alpha_8 = \beta_1 = \beta_3 = \gamma_1 = \gamma_3 = \delta = \epsilon = 0$. Since F_1 does not involve z_1, z_3 or z_3' all the elements in the first line except the first one are zero. Hence it is only necessary to consider the first diagonal element and its co-factor. In computing the elements of the determinant it is necessary to eliminate the coördinates and derivatives with the subscript two by means of the center of gravity equations. Then it is found from the explicit forms of the integrals and the transformations (23) that

$$\frac{\partial F_1}{\partial v_8} = m_1(1 + 2\omega a_7)(\xi_1^{(0)} - \xi_2^{(0)}) + m_3(2\omega b_7 + c_7)(\xi_3^{(0)} - \xi_2^{(0)}),$$

$$\frac{\partial F_2}{\partial \zeta_3} = -\, m_3\omega(\xi_3^{(0)} - \xi_2^{(0)}) \cos\frac{2\pi\omega}{\nu},$$

$$\frac{\partial F_2}{\partial \zeta_1'} = m_1(\xi_1^{(0)} - \xi_2^{(0)}) \sin\frac{2\pi\omega}{\nu},$$

$$\frac{\partial F_2}{\partial \zeta_3'} = m_3(\xi_3^{(0)} - \xi_2^{(0)}) \sin\frac{2\pi\omega}{\nu},$$

$$\frac{\partial F_3}{\partial \zeta_3} = -\, m_3\omega(\xi_3^{(0)} - \xi_2^{(0)}) \sin\frac{2\pi\omega}{\nu},$$

$$\frac{\partial F_3}{\partial \zeta_1'} = -\, m_1(\xi_1^{(0)} - \xi_2^{(0)}) \cos\frac{2\pi\omega}{\nu},$$

$$\frac{\partial F_3}{\partial \zeta_3'} = -\, m_3(\xi_3^{(0)} - \xi_2^{(0)}) \cos\frac{2\pi\omega}{\nu},$$

$$\frac{\partial F_4}{\partial \zeta_3} = 0,$$

$$\frac{\partial F_4}{\partial \zeta_1'} = \frac{m_1}{m_2}\nu[(m_1 + m_2)M_1 + m_3M_3],$$

$$\frac{\partial F_4}{\partial \zeta_3'} = \frac{m_3}{m_2}\nu[m_1M_1 + (m_2 + m_3)M_3],$$

When these elements are substituted in (32) and the determinant is reduced by the elementary rules for the simplification of determinants it is found that

$$D = -\frac{\partial F_1}{\partial \nu_8}\frac{m_1m_3^2(m_1 + m_2 + m)_3}{m_2}\omega\nu(\xi_3^{(0)} - \xi_2^{(0)})(M_3\xi_1^{(0)} - M_1\xi_3^{(0)}).$$

The factors of this expression are certainly all distinct from zero except the first and last two, which must be considered further. There is no loss of generality in so choosing the notation that $\xi_1^{(0)} < \xi_2^{(0)} < \xi_3^{(0)}$. Then it follows from the center of gravity equation

$$m_1\xi_1^{(0)} + m_2\xi_2^{(0)} + m_3\xi_3^{(0)} = 0$$

that $\xi_1^{(0)}$ is necessarily negative, $\xi_3^{(0)}$ is necessarily positive, and the next to the last factor of D is positive. It was remarked in (13) that $\omega^2 < A_3$. It follows from (9) that $\nu^2 + \omega^2 = A_1 + A_3$. Therefore $\nu^2 - A_1 > 0$ and M_3 has the sign of M_1/B_1. It follows from the definition of B_1 that it is positive. Hence both terms in the last factor of D have the same sign and this factor also is distinct from zero.

It only remains to consider $\partial F_1/\partial \nu_8$. It is found from equations (16)

and (21) that

$$a_7 = \frac{4\omega(\omega^2 - A_1) - 2\omega(\omega^2 + 2A_3)}{(\omega^2 + 2A_1)(\omega^2 + 2A_3) - 4B_1B_3} \equiv -\frac{2}{3\omega},$$

$$b_7 = \frac{-2\omega(\omega^2 + 2A_1)(\omega^2 - A_1) + 4\omega B_1 B_3}{B_1[(\omega^2 + 2A_1)(\omega^2 + 2A_3) - 4B_1B_3]} \equiv -\frac{2}{3\omega}\frac{\xi_3^{(0)}}{\xi_1^{(0)}},$$

$$c_7 = \frac{\omega^2 - A_1}{B_1} = \frac{B_3}{\omega^2 - A_3} = \frac{\xi_3^{(0)}}{\xi_1^{(0)}}.$$

When these expressions are simplified it follows that

$$1 + 2\omega a_7 = -\frac{1}{3}, \qquad 2\omega b_7 + c_7 = -\frac{3\xi_3^{(0)}}{\xi_1^{(0)}},$$

and therefore that both terms of $\partial F_1/\partial \nu_8$ are positive. Hence D is distinct from zero.

Since the four integrals can be solved at $t = 2\pi/\nu$ for ν_8, ζ_3, ζ_1' and ζ_3' as power series in $v_1 \cdots v_8$, $\alpha_1 \cdots \alpha_8$, β_1, β_3, γ_1, γ_3 vanishing for $v_1 = \cdots v_7 = \zeta_1 = 0$, the conditions

$$u_8\left(\frac{2\pi}{\nu}\right) - u_8 = 0, \quad z_3\left(\frac{2\pi}{\nu}\right) - z_3(0), \quad z_i'\left(\frac{2\pi}{\nu}\right) - z_i'(0) = 0 \quad (i = 1, 3)$$

are a consequence of the remainder of (29), and therefore can be suppressed. The periodicity conditions are then

$$\left.\begin{aligned}
0 &= \alpha_1[e^{(1+\delta)\rho_1 iT} - 1] + \epsilon P_1(\alpha_j, \beta_k, \gamma_k, \delta, \epsilon) \quad \begin{pmatrix} j = 1 \cdots 8 \\ k = 1, 3 \end{pmatrix},\\
0 &= \alpha_2[e^{-(1+\delta)\rho_1 iT} - 1] + \epsilon P_2(\alpha_j, \beta_k, \gamma_k, \delta, \epsilon),\\
0 &= \alpha_3[e^{(1+\delta)\omega iT} - 1] + \epsilon P_3(\alpha_j, \beta_k, \gamma_k, \delta, \epsilon),\\
0 &= \alpha_4[e^{-(1+\delta)\omega iT} - 1] + \epsilon P_4(\alpha_j, \beta_k, \gamma_k, \delta, \epsilon),\\
0 &= \alpha_5[e^{(1+\delta)\rho_2 T} - 1) + \epsilon P_5(\alpha_j, \beta_k, \gamma_k, \delta, \epsilon),\\
0 &= \alpha_6[e^{-(1+\delta)\rho_2 T} - 1] + \epsilon P_6(\alpha_j, \beta_k, \gamma_k, \delta, \epsilon),\\
0 &= \alpha_8 T + \epsilon P_7(\alpha_j, \beta_k, \gamma_k, \delta, \epsilon),\\
0 &= z_1(2\pi/\nu) - z_1(0) = M_1 \sin 2\pi(1 + \delta) + Q(\alpha_j, \beta_k, \gamma_k, \delta, \epsilon),
\end{aligned}\right\} \quad (33)$$

where Q vanishes for $\beta_k = \gamma_k = \epsilon = 0$ whatever the α_j and δ may be.

The first seven equations of (33) can be uniquely solved for $\alpha_1 \cdots \alpha_6$ and α_8 as power series in α_7, β_1, β_3, γ_1, γ_3, δ and ϵ vanishing for these quantities equal to zero. Indeed, the solutions vanish for $\epsilon = 0$. Suppose the results are substituted in the last equation of (33); it will then become a

function of α_7, β_1, β_3, γ_1, γ_3, δ and ϵ, vanishing with these quantities. The coefficient of δ to the first power comes from the first term alone and is M_1, which has been taken distinct from zero. Therefore, this equation can be solved for δ as power series in α_7, β_k, γ_k and ϵ, vanishing with these quantities. That is, equations (33) are uniquely solvable for $\alpha_1 \cdots \alpha_6$, α_8 and δ as power series in α_7, β_k, γ_k and ϵ, vanishing with these quantities. Therefore the periodic solutions having the period $2\pi/\nu$ exist.

Orthogonal Orbits with Period $2\pi/\nu$.—We have demonstrated the existence of a unique set of solutions with the period $2\pi/\nu$ by starting from general initial conditions. In the present article we shall show that all orbits of this type cross the x-axis at right angles. The conditions for an orthogonal start are $x_i' = y_i = z_i$ at $t = 0$, or

$$\left.\begin{aligned}
x_1' = 0 = {} & (1+\delta)\rho_1\iota a_1(\alpha_1+\alpha_2) + (1+\delta)\omega\iota a_3(\alpha_3+\alpha_4) \\
& \qquad + (1+\delta)\rho_2 a_5(\alpha_5+\alpha_6), \\
x_3' = 0 = {} & (1+\delta)\rho_1\iota b_1(\alpha_1+\alpha_2) + (1+\delta)w\iota b_3(\alpha_3+\alpha_4) \\
& \qquad + (1+\delta)\rho_2 b_5(\alpha_5+\alpha_6), \\
y_1 = 0 = {} & \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8, \\
y_3 = 0 = {} & c_1(\alpha_1+\alpha_2)+c_3(\alpha_3+\alpha_4)+c_5(\alpha_5+\alpha_6)+c_7(\alpha_7+\alpha_8), \\
z_1 = 0 = {} & M_{11}+M_{12}+M_{13}+M_{14}, \\
z_3 = 0 = {} & M_{31}+M_{32}+M_{33}+M_{34}.
\end{aligned}\right\} \quad (34)$$

The determinant of the $\alpha_i + \alpha_j$ of the right members of the first four equations cannot vanish for otherwise the determinant of equations (23) would vanish. Therefore the only solution of them is

$$\alpha_1+\alpha_2 = 0, \qquad \alpha_3+\alpha_4 = 0, \qquad \alpha_5+\alpha_6 = 0, \qquad \alpha_7+\alpha_8 = 0.$$

Similarly, since M_{3j} are expressible in terms of M_{1j} it follows that

$$M_1 = -M_{12}, \qquad M_{13} = -M_{14}.$$

Put

$$\alpha_1 = -\alpha_2 = \alpha', \quad \alpha_3 = -\alpha_4 = \alpha'', \quad \alpha_5 = -\alpha_6 = \alpha''', \quad \alpha_7 = -\alpha_8 = \alpha^{\text{iv}}.$$

By the symmetry theorem the conditions that the orbits be periodic are

$$x_i' = y_i = z_i \quad \text{at} \quad t = \frac{\pi}{\nu}\,.$$

These conditions are explicitly

$$\left.\begin{aligned}
0 &= 2(1+\delta)\rho_1\iota a_1\alpha' \sin \rho_1 \frac{\pi}{\nu}(1+\delta) + 2(1+\delta)\omega\iota a_3\alpha'' \sin \omega \frac{\pi}{\nu}(1+\delta) \\
&\quad + 2(1+\delta)\rho_2 a_5\alpha''' \sinh \rho_2 \frac{\pi}{\nu}(1+\delta) + \epsilon R_1(\alpha', \alpha'', \alpha''', \alpha^{\text{iv}}, \gamma_i, \delta, \epsilon), \\
0 &= 2(1+\delta)\rho_1\iota b_1\alpha' \sin \rho_1 \frac{\pi}{\nu}(1+\delta) + 2(1+\delta)\omega\iota b_3\alpha'' \sin \omega \frac{\pi}{\nu}(1+\delta) \\
&\quad + 2(1+\delta)\rho_2 b_5\alpha''' \sinh \rho_2 \frac{\pi}{\nu}(1+\delta) + \epsilon R_2(\alpha', \alpha'', \alpha''', \alpha^{\text{iv}}, \gamma_i, \delta, \epsilon), \\
0 &= 2\alpha' \sin \rho_1 \frac{\pi}{\nu}(1+\delta) + 2\alpha'' \sin \omega \frac{\pi}{\nu}(1+\delta) + 2\alpha''' \sinh \rho_2 \frac{\pi}{\nu}(1+\delta) \\
&\qquad\qquad + 2\alpha^{\text{iv}} + \epsilon R_3(\alpha', \alpha'', \alpha''', \alpha^{\text{iv}}, \gamma_i, \delta, \epsilon), \\
0 &= 2c_1\alpha' \sin \rho_1 \frac{\pi}{\nu}(1+\delta) + 2c_3\alpha'' \sin \omega \frac{\pi}{\nu}(1+\delta) \\
&\quad + 2c_5\alpha''' \sinh \rho_2 \frac{\pi}{\nu}(1+\delta) + 2c_7\alpha^{\text{iv}} + cR_4(\alpha', \alpha'', \alpha''', \alpha^{\text{iv}}, \gamma_i, \delta, \epsilon), \\
0 &= 2M_{11} \sin \omega \frac{\pi}{\nu}(1+\delta) + 2M_{13} \sin \pi(1+\delta) \\
&\qquad\qquad + \epsilon R_5(\alpha', \alpha'', \alpha''', \alpha^{\text{iv}}, \gamma_i, \delta, \epsilon), \\
0 &= 2M_{31} \sin \omega \frac{\pi}{\nu}(1+\delta) + 2M_{33} \sin \pi(1+\delta) \\
&\qquad\qquad + \epsilon R_6(\alpha', \alpha'', \alpha''', \alpha^{\text{iv}}, \gamma_i, \delta, \epsilon).
\end{aligned}\right\} \quad (35)$$

There are six equations and eight parameters α', α'', α''', α^{iv}, γ_i, δ, ϵ. Therefore two of them will remain undetermined when (35) are solved. We choose γ and ϵ arbitrarily. The determinant of the linear terms in α', α'', α''', α^{iv} is different from the determinant of equations (34) only by having $16 \sin \rho_1(\pi/\nu)(1+\delta) \sin \omega(\pi/\nu)(1+\delta) \sinh \rho_2(\pi/\nu)(1+\delta)$ as a factor. Since ρ_1, ω, ρ_2 and ν are by hypothesis incommensurable, the determinant does not vanish. We can therefore solve the first four equations uniquely for α', α'', α''', α^{iv} as power series in γ_i, δ, ϵ. Let the results of these solutions be substituted in the last two equations. The determinant of the linear terms in γ_3, δ is

$$\sin \omega \frac{\pi}{\nu} \begin{vmatrix} \dfrac{\nu B_1}{\omega\iota(\omega^2-\nu^2)}, & -\pi M_1 \\ -\dfrac{B_1}{(\omega^2-\nu^2)\iota}, & -\dfrac{\pi M_1(\nu^2 - A_1)}{B_1} \end{vmatrix}$$

which simplifies to

$$- (\nu^2 - A_1) \frac{(\nu + \omega B_1)\pi M_1 \sin \omega \dfrac{\pi}{\nu}}{\omega(\omega^2 - \nu^2)\iota}.$$

Since ω and ν are incommensurable this cannot vanish. Therefore we can solve the last two equations uniquely for γ_3 and δ as power series in γ_1 and ϵ. For any particular set of values of γ_1 and ϵ there is only one general solution and only one orthogonal solution. Therefore all solutions are orthogonal.

Interpretation of the Arbitrary Constants of the Solution.—The five initial constants α_7, β_1, β_3, γ_1, γ_3 and ϵ were chosen arbitrarily. The ϵ from the way it was introduced determines the magnitude of the deviations from the circular orbits; that is, the relative scale. It is properly called the relative scale, for whenever an orbit is found with a definite value of ϵ there is an infinity of others of different dimensions of the same general shape and the same properties.

The absolute scale of the circular orbits from which the bodies deviate is arbitrary. It is obvious, therefore, that deviations from given circular orbits can be made in such a way that the solutions still remain circular. That is, the final solutions depend on one parameter which is involved in the determination of the absolute scale of the orbits.

If the coordinate axes are not so chosen that the line of the bodies is the x-axis and their plane the xy-plane, then it is always possible by properly determining three constants to rotate to this position. Therefore three of the initial constants account for the position of the coördinate axes.

We are free to choose the origin of time, therefore another of the initial constants is determined by its choice.

To sum up, for general initial conditions there are six arbitrary parameters: two connected with the position of the plane of the bodies, one with the position of their line in this plane, one with the origin of time, and one each with the relative and absolute scales.

If the orbits existed only in the plane of initial motion then the two constants going with the position of this plane would not enter. In the orthogonal existence proofs we have determined all these constants except the relative and absolute scales. A similar interpretation of the arbitraries can be made in all the following cases:

Case 2. *Existence of Orbits with Period* $2\pi/\omega$.—We will now consider the analytic continuation of

$$x_i = K_i \cos \omega\tau, \qquad y_i = L_i \sin \omega\tau, \qquad z_i = M_i \sin \omega\tau,$$

or in the normal variables

$$u_i = 0 \quad (i = 1, 2, 5, 6, 7, 8), \qquad u_3 = ae^{\omega\iota\tau}, \qquad u_4 = ae^{-\omega\iota\tau},$$
$$z_i = M_i \sin \omega\tau \quad (i = 1, 3).$$

Let the initial conditions be

$$u_i = \alpha_i \quad (i = 1, 2, 5, 6, 7, 8), \qquad u_3 = a + \alpha_3, \qquad u_4 = a + \alpha_4,$$
$$z_i = \beta_i, \qquad z_i' = c_i + \gamma_i, \qquad c_i = \omega M_i \quad (i = 1, 3).$$

Integrate the equations of motion as power series in ϵ, introducing α_i, β_i, γ_i, δ as in the previous case. The solutions are exactly the same as equations (28) except the u_3 and u_4 equations which are:

$$u_3 = (a + \alpha_3)e^{(1+\delta)\omega\iota\tau} + \epsilon P_3(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),$$
$$u_4 = (a + \alpha_4)e^{-(1+\delta)\omega\iota\tau} + \epsilon P_4(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon).$$

Sufficient conditions that the orbits shall be periodic with period $2\pi/\omega$ are

$$u_i\left(\frac{2\pi}{\omega}\right) - u_i(0) = 0 \quad (i = 1 \cdots 8),$$

$$z_i\left(\frac{2\pi}{\omega}\right) - z_i(0) = 0, \qquad z_i'\left(\frac{2\pi}{\omega}\right) - z_i'(0) = 0 \quad (i = 1, 3).$$

Four of these periodicity equations are redundant. The argument is essentially the same as in par. 10. The explicit forms of the integrals are given by equations (30). We propose to show that the equations coming from u_8, u_3, z_3 and z_3' can be suppressed by means of these integrals. Let

$$u_i = \alpha_i + \nu_i \quad (i = 1, 2, 5, 6, 7, 8), \qquad u_3 = (a + \alpha_3)e^{\omega\iota\tau} + \nu_3,$$
$$u_4 = (a + \alpha_4)e^{-\omega\iota\tau} + \nu_4, \qquad z_i = (M_i + \gamma_i) \sin \omega\tau + \zeta_i,$$
$$z_i' = \omega(M_i + \gamma_i) \cos \omega\tau + \zeta_i' \quad (i = 1, 3),$$

where $M_3 = \xi_3^{(0)}/\xi_1^{(0)}M_1$ and $M_1 \neq 0$, $a \neq 0$. Following the argument of par. 10 the equations indicated can be suppressed provided the determinant

$$D = \begin{vmatrix} \dfrac{\partial F_1}{\partial \nu_8}, & \dfrac{\partial F_1}{\partial \nu_3}, & \dfrac{\partial F_1}{\partial \zeta_3}, & \dfrac{\partial F_1}{\partial \zeta_3'} \\ \dfrac{\partial F_2}{\partial \nu_8}, & \dfrac{\partial F_2}{\partial \nu_3}, & \dfrac{\partial F_2}{\partial \zeta_3}, & \dfrac{\partial F_2}{\partial \zeta_3'} \\ \dfrac{\partial F_3}{\partial \nu_8}, & \dfrac{\partial F_3}{\partial \nu_3}, & \dfrac{\partial F_3}{\partial \zeta_3}, & \dfrac{\partial F_3}{\partial \zeta_3'} \\ \dfrac{\partial F_4}{\partial \nu_8}, & \dfrac{\partial F_4}{\partial \nu_3}, & \dfrac{\partial F_4}{\partial \zeta_3}, & \dfrac{\partial F_4}{\partial \zeta_3'} \end{vmatrix} \neq 0.$$

The values of the elements of this determinant computed from equations

(30), (31) and (38) are as follows:

$$\frac{\partial F_1}{\partial \zeta_3} = \frac{\partial F_1}{\partial \zeta_3'} = \frac{\partial F_2}{\partial \zeta_3'} = \frac{\partial F_3}{\partial \zeta_3} = \frac{\partial F_3}{\partial \nu_8} = \frac{\partial F_3}{\partial \nu_3} = \frac{\partial F_4}{\partial \zeta_3'} = 0,$$

$$\frac{\partial F_3}{\partial \zeta_3'} = -\, m_3(\xi_3^{(0)} - \xi_2^{(0)}), \qquad \frac{\partial F_2}{\partial \zeta_3} = -\, \omega m_3(\xi_3^{(0)} - \xi_2^{(0)}),$$

$$\frac{\partial F_1}{\partial \nu_8} = m_1(\xi_1^{(0)} - \xi_2^{(0)})(1 + 2\omega a_7) + m_3(\xi_3^{(0)} - \xi_2^{(0)})(c_7 + 2\omega b_7),$$

$$\frac{\partial F_1}{\partial \nu_3} = m_1(\xi_1^{(0)} - \xi_2^{(0)})(\omega\iota + 2\omega a_3) + m_3(\xi_3^{(0)} - \xi_2^{(0)})(\omega\iota c_3 + 2\omega b_3),$$

$$\frac{\partial F_4}{\partial \nu_8} = 2\omega^2 m_1(\xi_1^{(0)} - \xi_2^{(0)})a_7 + 2\omega^2 m_3(\xi_3^{(0)} - \xi_2^{(0)})b_7,$$

$$\frac{\partial F_4}{\partial \nu_3} = 2\omega^2 m_1(\xi_1^{(0)} - \xi_2^{(0)})a_3 + 2\omega^2 m_3(\xi_3^{(0)} - \xi_2^{(0)})b_3.$$

Therefore D reduces to

$$\frac{\partial F_2}{\partial \zeta_3} \cdot \frac{\partial F_3}{\partial \zeta_3'} \begin{vmatrix} \dfrac{\partial F_1}{\partial \nu_8}, & \dfrac{\partial F_1}{\partial \nu_3} \\ \dfrac{\partial F_4}{\partial \nu_8}, & \dfrac{\partial F_4}{\partial \nu_3} \end{vmatrix}.$$

Neither of the first two factors vanish. The third factor easily reduces to

$$\begin{vmatrix} m_1(\xi_1^{(0)}-\xi_2^{(0)})+m_3(\xi_3^{(0)}-\xi_2^{(0)})c_7, & m_1(\xi_1^{(0)}-\xi_2^{(0)})\omega\iota+m_3(\xi_3^{(0)}-\xi_2^{(0)})\omega\iota c_3, \\ m_1(\xi_1^{(0)}-\xi_2^{(0)})a_7+m_3(\xi_3^{(0)}-\xi_2^{(0)})b_7, & m_1(\xi_1^{(0)}-\xi_2^{(0)})a_3+m_3(\xi_3^{(0)}-\xi_2^{(0)})b_7. \end{vmatrix} \tag{40}$$

The values of a_7, b_7 and c_7 are given by equations (32). We give below the values a_1, b_1 and c_1, as obtained from equations (16), noting that a_3, b_3 and c_3 can be obtained from a_1, b_1, c_1 by replacing $\rho_1\iota$ by $\omega\iota$

$$\left.\begin{aligned} a_1 &= \frac{-\,2\rho_1\omega B_1\iota(\rho_1^2 - \omega^2 - 2A_1 - 2A_3)}{\begin{aligned}(\rho_1^2 + \omega^2 + 2A_1)(\rho_1^2 + \omega^2 + 2A_3)(-\rho_1^2 - \omega^2 + A_1) \\ + 4\rho_1^2\omega^2(\rho_1^2 + \omega^2 + 2A_3) + 4B_1B_3(\rho_1^2 + \omega^2 + A_1)\end{aligned}}, \\ b_1 &= \frac{2\rho_1\omega\iota\{(\rho_1^2 + \omega^2 + 2A_1)(\rho_1^2 + \omega^2 - A_1) - 2B_1B_3 - 4\rho_1^2\omega^2\}}{d}, \\ c_1 &= \frac{-\,B_1(\rho_1^2 + \omega^2 + 2A_1)(\rho_1^2 + \omega^2 + 2A_3) + 8\rho^2\omega^2B_1 + 4B_1^2B_3}{d}, \end{aligned}\right\} \tag{41}$$

where d represents the denominator of a_1. From these values one finds by a short calculation that

$$b_3 = \frac{\xi_1^{(0)}}{\xi_3^{(0)}}a_3, \qquad c_3 = \frac{\xi_1^{(0)}}{\xi_3^{(0)}}, \qquad a_3 = -\frac{\iota}{2}\frac{\xi_3^{(0)}}{\xi_1^{(0)}},$$

while from equations (32)

$$b_7 = \frac{\xi_1^{(0)}}{\xi_3^{(0)}} a_7.$$

Equation (40) then reduces to

$$\left\{ m_1(\xi_1^{(0)} - \xi_2^{(0)}) + m_3(\xi_3^{(0)} - \xi_2^{(0)}) \frac{\xi_1^{(0)}}{\xi_3^{(0)}} \right\} \begin{vmatrix} 1 & \omega\iota \\ a_7 & a_3 \end{vmatrix}.$$

The first factor does not vanish since every term is negative. We have previously found

$$1 + 2\omega a_7 = -1/3 \qquad \text{or} \qquad 2\omega a_7 = -4/3.$$

Therefore the second factor reduces to

$$-\frac{1}{2}\left[\frac{\xi_3^{(0)}}{\xi_1^{(0)}} - \frac{4}{3}\right].$$

It follows that the determinant (39) does not vanish and the u_3, u_8, z_3, z_3' equations can be suppressed.

The necessary and sufficient conditions for a solution of period $2\pi/\omega$ are therefore

$$\left.\begin{aligned}
0 &= \alpha_1(e^{(1+\delta)\rho_1\iota(2\pi/\omega)} - 1) + \epsilon P_1(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
0 &= \alpha_2(e^{-(1+\delta)\rho_1\iota(2\pi/\omega)} - 1) + \epsilon P_2(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
0 &= (a + \alpha_4)(e^{-(1+\delta)\iota 2\pi} - 1) + \epsilon P_4(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
0 &= \alpha_5(e^{(1+\delta)\rho_2(2\pi/\omega)} - 1) + \epsilon P_5(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
0 &= \alpha_6(e^{-(1+\delta)\rho_2(2\pi/\omega)} - 1) + \epsilon P_6(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
0 &= \alpha_8\frac{2\pi}{\omega} + \epsilon P_7(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
0 &= M_{11}(e^{(1+\delta)2\pi\iota} - 1) + M_{12}(e^{-(1+\delta)2\pi\iota} - 1) + M_{13}(e^{(1+\delta)\nu\iota(2\pi/\omega)} - 1)\\
&\qquad + M_{14}(e^{-(1+\delta)\nu\iota(2\pi/\omega)} - 1) + \epsilon Q(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
0 &= \omega M_{11}(e^{(1+\delta)2\pi\iota} - 1) - \omega M_{12}(e^{-(1+\delta)2\pi\iota} - 1) + \nu M_{13}(e^{(1+\delta)\nu\iota(2\pi/\omega)} - 1)\\
&\qquad - \nu M_{14}(e^{-(1+\delta)\nu\iota(2\pi/\omega)} - 1) + \epsilon Q'(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),
\end{aligned}\right\} \quad (42)$$

where M_{i1}, M_{i2}, M_{i3}, M_{i4} are expressed in terms of β_i and γ_i by the equations

$$\left.\begin{aligned}
\beta_1 &= M_{11} + M_{12} + M_{13} + M_{14},\\
\beta_3 &= \frac{\omega^2 - A_1}{B_1}(M_{11} + M_{12}) + \frac{\nu^2 - A_1}{B_1}(M_{13} + M_{14}),\\
c_1 + \gamma_1 &= \omega\iota(M_{11} - M_{12}) + \nu\iota(M_{13} - M_{14}),\\
c_3 + \gamma_3 &= \omega\iota\frac{\omega^2 - A_1}{B_1}(M_{11} - M_{12}) + \nu\iota\frac{(\nu^2 - A_1)}{B_1}(M_{13} - M_{14}).
\end{aligned}\right\} \quad (43)$$

The first six of equations (42) can be solved uniquely for $\alpha_1, \alpha_2, \delta, \alpha_5, \alpha_6, \alpha_8$ as power series in $\epsilon, \alpha_3, \alpha_4, \alpha_7, \beta_i, \gamma_i$ since the determinant of the linear terms in these quantities does not vanish. Suppose these solutions are substituted in the last two equations. Then the determinant of the linear terms in β_1, γ_1 reduces to

$$-4\left(\frac{\nu^2 - A_1}{\nu^2 - \omega^2}\right)^2 \left(e^{\frac{\pi \nu \iota}{\omega}} - e^{-\frac{\pi \nu \iota}{\omega}}\right)^2.$$

Since ω and ν are incommensurable by hypothesis, it follows that this expression does not vanish. Therefore the last two equations can be solved uniquely for β_1 and γ_1, as power series in $\epsilon, \alpha_3, \alpha_4, \alpha_7, \beta_3, \gamma_3$. Thus the existence of an unique set of orbits with period $2\pi/\omega$ is proven. It can be shown that there is an unique set of period $2K(\pi/\omega)$ which include the case $K = 1$. It follows that all orbits of this type are reëntrant after one revolution. Further it can be shown, as in case 1, that a unique set of orthogonal orbits exists. It follows that the orthogonal orbits are the only ones.

Case 3. *Existence of Orbits with the Period* $2\pi/\rho_1$.—Let us consider the analytic continuation of the generating solutions

$$x_i = K_i \cos \rho_1 \tau, \qquad y_i = L_i \sin \rho_1 \tau, \qquad z_i = 0;$$

or, in the normal variables, of

$$u_1 = a e^{\rho_1 \iota \tau}, \qquad u_2 = a e^{-\rho_1 \iota \tau}, \qquad u_i = 0 \quad (i = 3, 4, 5, 6, 7, 8),$$
$$z_i = 0 \quad (i = 1, 3).$$

Let the initial conditions be

$$u_1 = a + \alpha_1, \qquad u_2 = a + \alpha_2, \qquad u_i = \alpha_i \quad (i = 3, 4, 5, 6, 7, 8),$$
$$z_i = \beta_i, \qquad z_i' = \gamma_i \quad (i = 1, 3).$$

The solutions of the differential equations with these initial conditions are

$$u_1 = (a + \alpha_1) e^{(1+\delta)\rho_1 \iota \tau} + \epsilon P_1(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),$$
$$u_2 = (a + \alpha_2) e^{-(1+\delta)\rho_1 \iota \tau} + \epsilon P_2(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),$$
$$u_3 = \alpha_3 e^{(1+\delta)\omega \iota \tau} + \epsilon P_3(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),$$
$$u_4 = \alpha_4 e^{-(1+\delta)\omega \iota \tau} + \epsilon P_4(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),$$
$$u_5 = \alpha_5 e^{(1+\delta)\rho_2 \tau} + \epsilon P_5(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),$$
$$u_6 = \alpha_6 e^{-(1+\delta)\rho_2 \tau} + \epsilon P_6(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),$$
$$u_7 = \alpha_7 + \alpha_8 \tau + \epsilon P_7(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),$$
$$u_8 = \alpha_8 + \epsilon P_8(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),$$

$$z_i = M_{i1}e^{(1+\delta)\omega\iota\tau} + M_{i2}e^{-(1+\delta)\omega\iota\tau} + M_{i3}e^{(1+\delta)\nu\iota\tau} + M_{i4}e^{-(1+\delta)\nu\iota\tau} + \epsilon Q_i(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),$$

$$z'_i = \{\omega M_{i1}e^{(1+\delta)\omega\iota\tau} - \omega M_{i2}e^{-(1+\delta)\omega\iota\tau} + \nu M_{i3}e^{(1+\delta)\nu\iota\tau} - \nu M_{i4}e^{-(1+\delta)\nu\iota\tau}\}(1+\delta)\iota + \epsilon Q'_i(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon).$$

The M_{ij} are homogeneous and linear in the β_i, γ_i. It will be proved that the analytic continuation of this solution exists only in the xy-plane.

The periodicity conditions are of the same form as in the other two cases. Without writing them out in full, it is clear that if $\beta_i = \gamma_i = 0$ the orbits are wholly in the xy-plane. On the other hand, if β_i, γ_i are different from zero, we shall show that the solution is impossible. The linear terms of the z_i-equations are homogeneous in β_i, γ_i with a determinant different from zero. We solve three of them for β_1, β_3 and γ_1 and substitute the results in the fourth equation. Then γ_3 is a factor and can be divided out. There is a term left independent of all the other quantities α_i, δ_i, ϵ. Therefore the solution as a power series in the remaining parameters, vanishing with them, is impossible.

The conditions for periodicity become therefore

$$u_i\left(\frac{2\pi}{\rho_1}\right) - u_i(0) = 0 \quad (i = 1, \cdots, 8).$$

Again these conditions are not independent, but now only two of them are redundant. There are only two integrals when the problem is in the plane, the energy integral and one integral of areas. It can be shown as in case (1) that the u_2- and u_7-equations are consequences of the others.

The suppression of these two equations is possible provided the Jacobian

$$D = \begin{vmatrix} \dfrac{\partial F_1}{\partial \nu_8}, & \dfrac{\partial F_1}{\partial \nu_2} \\ \dfrac{\partial F_4}{\partial \nu_8}, & \dfrac{\partial F_4}{\partial \nu_2} \end{vmatrix}$$

is different from zero for zero values of the initial constants, δ and ϵ. From the explicit values of the integrals (equations 30) we find

$$\frac{\partial F_1}{\partial \nu_8} = m_1(\xi_1^{(0)} - \xi_2^{(0)})(2\omega a_7 + 1) + m_3(\xi_3^{(0)} - \xi_2^{(0)})(2\omega b_7 + c_7),$$

$$\frac{\partial F_1}{\partial \nu_2} = m_1(\xi_1^{(0)} - \xi_2^{(0)})(2\omega a_1 + \rho_1\iota) + m_3(\xi_3^{(0)} - \xi_2^{(0)})(2\omega b_1 + c_1\rho_1\iota),$$

$$\frac{\partial F_4}{\partial \nu_8} = \omega^2\{m_1(\xi_1^{(0)} - \xi_2^{(0)})a_7 + m_3(\xi_3^{(0)} - \xi_2^{(0)})b_7\},$$

$$\frac{\partial F_4}{\partial \nu_2} = \omega^2\{m_1(\xi_1^{(0)} - \xi_2^{(0)})a_1 + m_3(\xi_3^{(0)} - \xi_2^{(0)})b_1\}.$$

The values of $2\omega a_7 + 1$ and $2\omega b_7 + c_7$ are given in equations (32). Multiplying the first row of D by 2, subtracting the second row from it and expanding we obtain

$$D = -\frac{2\omega}{3}\left\{ m_1(\xi_1^{(0)} - \xi_2^{(0)}) + m_3(\xi_3^{(0)} - \xi_2^{(0)})\frac{\xi_3^{(0)}}{\xi_1^{(0)}} \right\} \{m_1(\xi_1^{(0)} - \xi_2^{(0)})(3\omega a_1 + \rho_1\iota) + m_3(\xi_3^{(0)} - \xi_2^{(0)})(3\omega b_1 + 2c_1\rho_1\iota)\}.$$

The first two factors are clearly negative. We shall prove that $3\omega a_1 + \rho_1\iota$ is opposite in sign to $3\omega b_1 + 2c_1\rho_1\iota$. The values of a_1, b_1 and c_1 are

$$a_1 = \frac{2\rho_1\omega\iota\{\rho_1^2 + \omega^2 - 2A_1 - 2A_3\}}{(\rho_1^2 + \omega^2 + 2A_1)(\rho_1^2 + \omega^2 + 2A_3) - 8\rho_1^2\omega^2 - 4B_1B_3} = \frac{E}{d},$$

$$b_1 = -\frac{2\rho_1\omega\iota\{(\rho_1^2 + \omega^2 + 2A_1)(\rho_1^2 + \omega^2 - A_1) - 4\rho_1^2\omega^2 - 2B_1B_3\}}{d} = \frac{F}{d},$$

$$c_1 = \frac{\begin{array}{l}(\rho_1^2 + \omega^2 + 2A_1)(\rho_1^2 + \omega^2 + 2A_3)(\rho_1^2 + \omega^2 - A_1)\\ \qquad - 4\rho_1^2\omega^2(\rho_1^2 + \omega^2 + 2A_3) - 4B_1B_3(\rho_1^2 + \omega^2 - A_1)\end{array}}{d} = \frac{G}{d}.$$

To determine the sign of $3\omega E + 2\rho_1\iota d$ we first divide out the factor $2\rho_1\iota$, then substitute for ρ_1^4 its value in terms of ρ_1^2 given by equation (12). After considerable simplification there results

$$\omega^4 - 5\omega^2(A_1 + A_3) + \rho_1^2(A_1 + A_3) + 4A_1A_3 + 2A_1^2 + 2A_3^2.$$

By substituting $\lambda^2 = -A_1$ and $\lambda^2 = -A_3$ in (12) we find $\rho_1^2 > A_1$ and $\rho_1^2 > A_3$. We have already proven $A_1 > \omega^2$ and $A_3 > \omega^2$. It follows that the last four terms are greater than $5\omega^2(A_1 + A_3)$, therefore $3\omega a_1 + \rho_1\iota$ is of the same sign as d. After eliminating ρ_1^4 by (12) and simplifying we find

$$b_1 = -\frac{2\rho_1\omega\iota(\omega^2 - A_3)(\rho_1^2 + \omega^2 - 2A_1 - 2A_3)}{d},$$

$$c_1 = \frac{3\rho_1^2\omega^2 - (A_1 + A_3)(\rho_1^2 + \omega^2 - 2A_1 - 2A_3)}{d}.$$

Then

$$3\omega b_1 + \rho_1\iota c_1 = \frac{\begin{array}{l}2\rho_1\iota(\omega^2 - A_3)[-3\omega^4 + 2\omega^2(A_1 + A_3)\\ \qquad + (A_1 + A_3)(-\rho_1^2 + 3\omega^2 + 2A_1 + 2A_3)]\end{array}}{d}.$$

It follows that $3\omega b_1 + \rho_1\iota c_1$ has the sign opposite to that of d. Since $\xi_1^{(0)} - \xi_2^{(0)}$ is negative and $\xi_3^{(0)} - \xi_2^{(0)}$ is positive, all the terms in the third factor of D are of the same sign. Therefore the u_2 and u_8 equations can be suppressed.

Necessary and sufficient conditions that the orbits shall be periodic with the period $2\pi/\rho_1$ become

$$\left.\begin{aligned}
0 &= (a+\alpha_1)(e^{(1+\delta)\iota 2\pi}-1)+\epsilon P_1(\alpha_i,\delta,\epsilon),\\
0 &= \alpha_3(e^{(1+\delta)\omega\iota(2\pi/\rho_1)}-1)+\epsilon P_3(\alpha_i,\delta,\epsilon),\\
0 &= \alpha_4(e^{-(1+\delta)\omega\iota(2\pi/\rho_1)}-1)+\epsilon P_4(\alpha_i,\delta,\epsilon),\\
0 &= \alpha_5(e^{(1+\delta)\rho_2(2\pi/\rho_1)}-1)+\epsilon P_5(\alpha_i,\delta,\epsilon),\\
0 &= \alpha_6(e^{-(1+\delta)\rho_2(2\pi/\rho_1)}-1)+\epsilon P_6(\alpha_i,\delta,\epsilon),\\
0 &= \alpha_8(2\pi/\rho_1)+\epsilon P_8(\alpha_i,\delta,\epsilon).
\end{aligned}\right\} \tag{45}$$

There are six equations and the ten parameters, $\alpha_1 \cdots \alpha_8, \delta, \epsilon$. Therefore we can choose $\alpha_1, \alpha_2, \alpha_8, \epsilon$ arbitrarily and solve for the others as power series in $\alpha_1, \alpha_2, \alpha_8, \epsilon$ vanishing with these arguments. The equations have unique solutions since the determinant of the linear terms in $\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \delta$ is

$$a(e^{(1+\delta)\omega\iota(2\pi/\rho_1)}-1)(e^{-(1+\delta)\omega\iota(2\pi/\rho_1)}-1)(e^{(1+\delta)\rho_2(2\pi/\rho_1)}-1)(e^{-(1+\delta)\rho_2(2\pi/\rho_1)}-1)\frac{2\pi}{\rho_1}\neq 0$$

and since $\alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \delta = 0$ is not a solution. This proves the existence of a unique set of orbits in the plane with the period $2\pi/\rho_1$.

We can show, as in cases (1) and (2), that all orbits of this type are reëntrant after one revolution, and are orthogonal.

Cases 4, 6 *and* 7.—The generating solution for these orbits is

$$x_i = K_i \cos\omega\tau, \qquad y_i = L_i \sin\omega\tau, \qquad z_i = M_i \sin\omega\tau + N_i \sin\nu\tau,$$

or, in the normal variables,

$$u_i = 0 \quad (i = 1, 2, 5, 6, 7, 8), \qquad u_3 = ae^{\omega\iota\tau}, \qquad u_4 = ae^{-\omega\iota\tau},$$
$$z_i = M_i \sin\omega\tau + N_i \sin\nu\tau.$$

Let the initial conditions be

$$u_i = \alpha_i \quad (i = 1, 2, 5, 6, 7, 8), \qquad u_3 = a + \alpha_3, \qquad u_4 = a + \alpha_4,$$
$$z_i = \beta_i, \qquad z_i' = c_i + \gamma_i \quad \text{where} \quad c_i = \omega M_i + \nu N_i.$$

With these initial conditions, the solutions of the differential equations (22) are

$$\begin{aligned}
u_1 &= \alpha_1 e^{(1+\delta)\rho_1\iota\tau} + \epsilon P_1(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_2 &= \alpha_2 e^{-(1+\delta)\rho_1\iota\tau} + \epsilon P_2(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_3 &= (a+\alpha_3)e^{(1+\delta)\omega\iota\tau} + \epsilon P_3(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_4 &= (a+\alpha_4)e^{-(1+\delta)\omega\iota\tau} + \epsilon P_4(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_5 &= \alpha_5 e^{(1+\delta)\rho_2\tau} + \epsilon P_5(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),
\end{aligned}$$

$$
\left.\begin{aligned}
u_6 &= \alpha_6 e^{-(1+\delta)\rho_3\tau} + \epsilon P_6(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_7 &= \alpha_8\tau + \alpha_7 + \epsilon P_7(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_8 &= \alpha_8 + \epsilon P_8(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
z_i &= M_{i1}e^{(1+\delta)\omega\iota\tau} + M_{i2}e^{-(1+\delta)\omega\iota\tau} + M_{i3}e^{(1+\delta)\nu\iota\tau} + M_{i4}e^{-(1+\delta)\nu\iota\tau}\\
&\qquad + \epsilon Q_i(\alpha_j, \beta_j, \gamma_j, \delta, \epsilon),\\
z_i' &= (1+\delta)\iota\{\omega M_{i1}e^{(1+\delta)\omega\iota\tau} - \omega M_{i2}e^{-(1+\delta)\omega\iota\tau} + \nu M_{i3}e^{(1+\delta)\nu\iota\tau}\\
&\qquad - \nu M_{i4}e^{-(1+\delta)\nu\iota\tau}\} + \epsilon Q_i'(\alpha_j, \beta_j, \gamma_j, \delta, \epsilon).
\end{aligned}\right\} \quad (46)
$$

The M_{ij} are the solutions of the equations

$$\beta_i = M_{11} + M_{12} + M_{13} + M_{14},$$

$$\beta_3 = \frac{\omega^2 - A_1}{B_1}(M_{11} + M_{12}) + \frac{\nu^2 - A_1}{B_1}(M_{13} + M_{14}),$$

$$\frac{c_1 + \gamma_1}{(1+\delta)\iota} = \omega(M_{11} - M_{12}) + \nu(M_{13} - M_{14}),$$

$$\frac{c_3 + \gamma_3}{(1+\delta)\iota} = \omega\frac{\omega^2 - A_1}{B_1}(M_{11} - M_{12}) + \nu\frac{\nu^2 - A_1}{B_1}(M_{13} - M_{14}),$$

where $c_1 = \omega M_1 + \nu N_1$, $c_3 = \omega M_3 + \nu N_3$. By equations (13) we have the relations

$$M_3 = \frac{\omega^2 - A_1}{B_1}M_1, \qquad N_3 = \frac{\nu^2 - A_1}{B_1}N_1.$$

Making use of these relations, it is found that the M_{ij} are non-homogeneous in β_i and γ_i.

The periodicity conditions

$$u_i(T) - u_i(0) = 0 \quad (i = 1 \cdots 8), \qquad T = \frac{2k\pi}{\omega} = \frac{2k'\pi}{\nu},$$

$$z_i(T) - z_i(0) = 0, \qquad z_i'(T) - z_i'(0) = 0 \quad (i = 1, 3),$$

are not independent, but we do not need to discuss their independence here. We can show that the orbits do not exist unless there are certain relations among the constants of the generating solution. Suppose we solve some of the u_i-equations for α_i, all the remaining α_i to be chosen arbitrarily. We substitute these solutions in the equation coming from u_3. In place of the arbitrary α_j put $\eta_j\epsilon$. It follows that the value of δ coming from the u_3-equation begins with terms in ϵ^2. We substitute this value in the equa-

tions

$$z_i(T) - z_i(0) = 0 \quad (i = 1, 3)$$

and divide out the factor ϵ from each equation. Then, in each equation there is left a term independent of all the initial constants. One might make these terms vanish by choosing the constants of the generating solution properly, but the computation has proved so complicated that a full discussion has not been made.

The same condition of affairs turns up in cases 6 and 7 and we are at once able to say that no orbits exist in these cases except, possibly, when there are special relations among the constants of the generating solution.

Case 5. *Orbits in which ρ_1 and ν are commensurable.*—We shall prove the existence of orbits of which the period T is the least common integral multiple of $2\pi/\rho_1$ and $2\pi/\nu$. The generating solution is

$$u_1 = ae^{\rho_1\iota\tau}, \qquad u_2 = ae^{-\rho_1\iota\tau}, \qquad u_i = 0 \quad (i = 3, 4, 5, 6, 7, 8),$$
$$z_i = M_i \sin \nu\tau \quad (i = 1, 3).$$

Let the initial conditions be

$$u_1 = a + \alpha_1, \qquad u_2 = a + \alpha_2, \qquad u_i = \alpha_i \quad (i = 3, 4, 5, 6, 7, 8),$$
$$z_i = \beta_i, \qquad z_i' = c_i + \gamma_i, \qquad c_i = \nu M_i \quad (i = 1, 3).$$

The solutions of the differential equations (22) with these initial conditions are

$$\begin{aligned}
u_1 &= (a + \alpha_1)e^{(1+\delta)\rho_1\iota\tau} + \epsilon P_1(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_2 &= (a + \alpha_2)e^{-(1+\delta)\iota\tau} + \epsilon P_2(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_3 &= \alpha_3 e^{(1+\delta)\omega\iota\tau} + \epsilon P_3(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_4 &= \alpha_4 e^{-(1+\delta)\omega\iota\tau} + \epsilon P_4(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_5 &= \alpha_5 e^{(1+\delta)\rho_2\tau} + \epsilon P_5(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_6 &= \alpha_6 e^{-(1+\delta)\rho_2\tau} + \epsilon P_6(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_7 &= \alpha_8\tau + \alpha_7 + \epsilon P_7(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_8 &= \alpha_8 + \epsilon P_8(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),
\end{aligned} \tag{47}$$

$$\begin{aligned}
z_i = M_{i1}e^{(1+\delta)\omega\iota\tau} + M_{i2}e^{-(1+\delta)\omega\iota\tau} + M_{i3}e^{(1+\delta)\nu\iota\tau} + M_{i4}e^{-(1+\delta)\nu\iota\tau}&\\
+ \epsilon Q_i(\alpha_j, \beta_j, \gamma_j, \delta, \epsilon)&,
\end{aligned}$$

$$\begin{aligned}
z_i' = (1 + \delta)\, \iota\{\omega M_{i1}e^{(1+\delta)\omega\iota\tau} - \omega M_{i2}e^{-(1+\delta)\omega\iota\tau} + \nu M_{i3}e^{(1+\delta)\nu\iota\tau}&\\
- \nu M_{i4}e^{-(1+\delta)\iota\tau}\} + \epsilon Q_i'(\alpha_j, \beta_j, \gamma_j, \delta, \epsilon)&,
\end{aligned}$$

where the M_{ij} have the same form as the expressions (25); that is, M_{11}, M_{12} are homogeneous in β_i, γ_i but M_{13}, M_{14} each have a term independent of β_i and γ_i. The periodicity conditions are of the same form as before.

The argument and method of proof is almost identical with case 2. The equations coming from u_8, u_2, z_3 and z_3' can be suppressed provided the Jacobian

$$D = \begin{vmatrix} \frac{\partial F_1}{\partial \nu_8}, & \frac{\partial F_1}{\partial \nu_2}, & \frac{\partial F_1}{\partial \xi_3}, & \frac{\partial F_1}{\partial \xi_3'} \\ \frac{\partial F_2}{\partial \nu_8}, & \frac{\partial F_2}{\partial \nu_2}, & \frac{\partial F_2}{\partial \xi_3}, & \frac{\partial F_2}{\partial \xi_3'} \\ \frac{\partial F_3}{\partial \nu_8}, & \frac{\partial F_3}{\partial \nu_2}, & \frac{\partial F_3}{\partial \xi_3}, & \frac{\partial F_3}{\partial \xi_3'} \\ \frac{\partial F_4}{\partial \nu_8}, & \frac{\partial F_4}{\partial \nu_2}, & \frac{\partial F_4}{\partial \xi_3}, & \frac{\partial F_4}{\partial \xi_3'} \end{vmatrix}$$

is distinct from zero.

The values of the elements of D which are needed are:

$$\frac{\partial F_1}{\partial \xi_3} = \frac{\partial F_1}{\partial \xi_3'} = \frac{\partial F_2}{\partial \nu_8} = \frac{\partial F_2}{\partial \xi_3'} = \frac{\partial F_3}{\partial \xi_3} = \frac{\partial F_4}{\partial \xi_3} = \frac{\partial F_4}{\partial \xi_3'} = 0,$$

$$\frac{\partial F_2}{\partial \xi_3} = -\,\omega m_3(\xi_3^{(0)} - \xi_2^{(0)}) \cos \frac{2k\pi\omega}{\nu},$$

$$\frac{\partial F_3}{\partial \xi_3'} = -\,m_3(\xi_3^{(0)} - \xi_2^{(0)}) \cos \frac{2k\pi\omega}{\nu},$$

$$\frac{\partial F_1}{\partial \nu_8} = m_1(\xi_1^{(0)} - \xi_2^{(0)})(1 + 2\omega a_7) + m_3(\xi_3^{(0)} - \xi_2^{(0)})(c_7 + 2\omega b_7),$$

$$\frac{\partial F_1}{\partial \nu_2} = m_1(\xi_1^{(0)} - \xi_2^{(0)})(\rho_1\iota + 2\omega a_1) + m_3(\xi_3^{(0)} - \xi_2^{(0)})(\rho_1\iota c_1 + 2\omega b_1),$$

$$\frac{\partial F_4}{\partial \nu_8} = \omega^2\{m_1(\xi_1^{(0)} - \xi_2^{(0)})a_7 + m_3(\xi_3^{(0)} - \xi_2^{(0)})b_7\},$$

$$\frac{\partial F_4}{\partial \nu_2} = \omega^2\{m_1(\xi_1^{(0)} - \xi_2^{(0)})a_1 + m_3(\xi_3^{(0)} - \xi_2^{(0)})b_1\}.$$

Therefore D reduces to

$$-\frac{\partial F_2}{\partial \xi_3} \cdot \frac{\partial F_3}{\partial \xi_3'} \begin{vmatrix} \frac{\partial F_1}{\partial \nu_8}, & \frac{\partial F_1}{\partial \nu_2} \\ \frac{\partial F_4}{\partial \nu_8}, & \frac{\partial F_4}{\partial \nu_2} \end{vmatrix}.$$

Neither of the first two factors vanish and the third factor has already been discussed in case 3. The remainder of the existence proof is identical with case 2.

Comparison of the Commensurable Cases with the Incommensurable.—In case 5 it has been found that the orbits exist for general values of the constants of the generating solution. Necessarily they still exist for particular values. Then in case 5 put $M_1 = 0$. It follows that $M_3 = 0$ and the generating solution becomes the same as in case 3. The conclusion is that the orbits of case 3 exist even when ρ_1 and ν are commensurable. If we put $a = 0$ in case 5, the generating solution becomes the same as in case 1. Hence the orbits of case 1 exist when ρ_1 and ν are commensurable. Cases 4, 6 and 7 have not been completely discussed and consequently the existence of the orbits in cases 1, 2 and 3, when ω and ν are commensurable, when ρ_1 and ω are commensurable, and when ρ_1, ω and ν are commensurable, is not proven.

ON CERTAIN CHAINS OF THEOREMS IN REFLEXIVE GEOMETRY.

By Flora D. Sutton.

§ 1. Introduction.

It is of interest to extend properties of the triangle to more than three lines and, when possible, to n lines of a plane. This can often be done by a process called geometrical interpolation. As an illustration, I consider, in what follows, a problem proposed by Desboves in his "Questions de Géométrie Élementaire," * namely: "Des trois sommets a, b, c d'un triangle on abaisse des perpendiculaires ap, bq, cr sur une droite quelconque de son plan, pius, des points p, q, r des perpendiculaires sur bc, ac, ab: ces trois dernières droites se coupent en un même point." A discussion of this problem, by means of trilinear coördinates, is given by Kazimierz Cwojdzinski in *Archiv der Mathematik und Physik*.†

In the present paper some extensions of this theorem are considered as indicated in the following scheme:

Undirected Lines.

First Chain.		Second Chain.	
1—*a*.	3 lines	2—*a*.	4 lines
1—*b*.	4 lines	2—*b*.	5 lines
1—*c*.	5 lines	2—*c*.	6 lines
1—*g*.	general statement	2—*g*.	general statement

Directed Lines.

I—*a*.	4 lines	II—*a*.	5 lines
I—*b*.	5 lines	II—*b*.	6 lines
I—*g*.	general statement	II—*g*.	general statement

We name a point of the plane by a complex number.

§ 2. Some Fundamental Formulæ in Reflexive Geometry.

1. *Reflexion in a Line.*

If the two triangles x, a, b and y, b, a are inversely similar, and y is the reflexion of x, b the reflexion of a, and a the reflexion of b, in one and the same line, then the two triangles x, a, b and $\bar{y}$, $\bar{b}$, $\bar{a}$ are directly similar.

* 1875; page 241; No. 77.

† 1901; Vol. 1; pp. 175–180.

The condition that any two triangles x_1, x_2, x_3 and y_1, y_2, y_3 be directly similar is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0; \tag{1}$$

therefore, since the two triangles x, a, b and $\bar{y}, \bar{b}, \bar{a}$ are directly similar,

$$\begin{vmatrix} x & a & b \\ \bar{y} & \bar{b} & \bar{a} \\ 1 & 1 & 1 \end{vmatrix} = 0. \tag{2}$$

Expanding this determinant, we obtain

$$\frac{x}{(a-b)} + \frac{\bar{y}}{(\bar{a}-\bar{b})} = \frac{a\bar{a} - b\bar{b}}{(a-b)(\bar{a}-\bar{b})}, \tag{3}$$

which can be written more symmetrically, as

$$\frac{x-b}{a-b} + \frac{\bar{y}-\bar{b}}{\bar{a}-\bar{b}} = 1 \tag{4}$$

or

$$\frac{x-a}{a-b} + \frac{\bar{y}-\bar{a}}{\bar{a}-\bar{b}} = 1. \tag{5}$$

If $b = 0 =$, $\bar{b}$ (4) becomes

$$\frac{x}{a} + \frac{\bar{y}}{\bar{a}} = 1, \tag{6}$$

which is the standard equation for reflexion in a line.

2. *Map-equation of the Double Parabola.**

The map-equation of the double parabola is

$$x = \frac{A_1}{(\alpha_1 - t)^2} + \frac{A_2}{(\alpha_2 - t)^2} \tag{1}$$

(where α_1, α_2, t are turns or orthogonal numbers), provided the cusp condition $dx/dt = 0$ is satisfied for t a turn.

Applying the cusp condition to (1) we have

$$\frac{dx}{dt} = \sum^{2} \frac{2A_i}{(\alpha_i - t)^3} = 0. \tag{2}$$

The conjugate equation of (2) is

$$\sum^{2} \frac{2\bar{A}_i\alpha_i^3 t^3}{(\alpha_i - t)^3} = 0. \tag{$\bar{2}$}$$

* The name is due to Clifford.

Hence, the condition that (1) has a cusp is satisfied when

$$A_i = \overline{A}_i \alpha_i^3. \tag{3}$$

And, in general, the map-equation of an n-fold parabola is

$$x = \sum^n \frac{A_i}{(\alpha_i - t)^2}, \tag{4}$$

provided the cusp condition $dx/dt = 0$ is satisfied for a turn t, that is, when (3) is satisfied for $i = 1, 2, 3, \cdots, n$.

3. *The Equation of a Directed Line.*

Directed lines are lines which possess a right- and left-hand side, and therefore should be marked with an arrow head.

The normal form of a line, in rectangular coördinates, is

$$X \cos\alpha + Y \sin\alpha = p, \tag{1}$$

where p is the $\perp$ distance from the line to the origin.

If, now, we employ the circular coördinates $x = X + iY$, $\bar{x} = X - iY$, instead of the rectangular coördinates, (1) becomes

$$(x + \bar{x}) \cos\alpha + \left(\frac{x - \bar{x}}{i}\right) \sin\alpha = 2p \tag{2}$$

or

$$x(\cos\alpha - i \sin\alpha) + \bar{x}(\cos\alpha + i \sin\alpha) = 2p. \tag{3}$$

But $(\cos\alpha - i \sin\alpha)$ is a turn; let us represent it by t, then

$$(\cos\alpha + i \sin\alpha) = 1/t.$$

Putting these values in (3) we have

$$xt + \bar{x}/t = 2p \tag{4}$$

as the equation of a directed line.

4. *Distance from a Straight Line.*

In the case of directed lines, distance = length ÷ direction.

Let 0 be the reflexion of $a = 2p/t$ (where p is real) in the axis, then the equation of the axis is

$$yt + \bar{x}/t = 2p. \tag{1}$$

From the figure we see that the distance of the point x from the axis is

$$\frac{x - y}{2 \cdot 1/t}. \tag{2}$$

But from (1) we have that

$$ty = -\bar{x}/t + 2p. \tag{3}$$

Therefore,

$$\frac{x-y}{2\cdot 1/t} = \tfrac{1}{2}(tx + \bar{x}/t - 2p). \tag{4}$$

Thus, twice the distance of a point x from a line is expressed by the equation

$$2D = tx + \bar{x}/t - 2p.$$

Undirected Lines.

§ 3. First Chain of Theorems.

1—*a*. Given three lines λ_1, λ_2 and λ_3 tangent to the parabola

$$x = \frac{1}{(1+t)^2} \tag{1}$$

at the points t_1, t_2 and t_3 respectively, then the equation of λ_i (for $i = 1, 2, 3$) will be

$$x = \frac{1}{(1+t_i)(1+t)}. \tag{2}$$

The intersection of λ_1 and λ_2 is given by the equation

$$x_{12} = \frac{1}{(1+t_1)(1+t_2)}, \tag{3}$$

for if we let $t = t_2$ in the equation of λ_1, it is identical with (3); therefore (3) is the equation of a point on line λ_1; and similarly, if we let $t = t_1$, in the equation of λ_2, this equation becomes the same as (3); therefore (3) is the equation of a point on the line λ_2, consequently (3) is the equation of the intersection of the two lines λ_1 and λ_2. In like manner, we obtain the intersections of λ_2 and λ_3, and λ_3 and λ_1. Its conjugate equation is

$$\bar{x}_{12} = \frac{t_1 t_2}{(1+t_1)(1+t_2)}. \tag{4}$$

We shall now proceed to reflect these intersections x_{12}, x_{23}, x_{31} in an arbitrary line, say the line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1. \tag{5}$$

From section 1 we see that reflexion in the line is given by the equation

$$\frac{x}{a} + \frac{\bar{y}}{\bar{a}} = 1. \tag{6}$$

Therefore, reflecting the point x_{12} (the intersection of λ_1 and λ_2) whose coördinates are

$$x_{12} = \frac{1}{(1+t_1)(1+t_2)}; \qquad \bar{x} = \frac{t_1 t_2}{(1+t_1)(1+t_2)}, \tag{7}$$

we obtain, as its reflexion in the line, the point

$$\frac{x}{a} = 1 - \frac{t_1 t_2}{\bar{a}(1+t_1)(1+t_2)}, \tag{8}$$

its conjugate

$$\frac{\bar{x}}{\bar{a}} = 1 - \frac{1}{a(1+t_1)(1+t_2)}. \tag{9}$$

In a like manner we can obtain the reflexions of the points x_{23} and x_{31} in the line.

The equation of a tangent to the parabola at the point t_1 is

$$x = \frac{1}{(1+t_1)(1+t)} \tag{10}$$

and its conjugate equation is

$$\bar{x} = \frac{t_1 t}{(1+t_1)(1+t)}. \tag{11}$$

Adding (10) and (11) we have

$$x + \bar{x} = \frac{1}{(1+t_1)(1+t)} + \frac{t_1 t}{(1+t_1)(1+t)}, \tag{12}$$

or

$$x + \frac{\bar{x}}{t_1} = \frac{1}{1+t_1}, \tag{13}$$

which is a self-conjugate equation of the tangent to the parabola at the point t_1. A line perpendicular to this tangent will be of the form

$$x - \frac{\bar{x}}{t_1} = c - \frac{\bar{c}}{t_1}. \tag{14}$$

Consequently, the equation of a line on the reflexion of x_{23} and perpendicular to the line λ_1 (which is tangent to the parabola at the point t_1) is

$$x - \frac{\bar{x}}{t_1} = a - \frac{a t_2 t_3}{\bar{a}(1+t_2)(1+t_3)} - \frac{1}{t_1}\left[\bar{a} - \frac{\bar{a}}{a(1+t_2)(1+t_3)}\right]. \tag{15}$$

For convenience let us rewrite (15) in the following manner:

$$t_1x - \bar{x} = at_1 - \frac{at_1t_2t_3(1 + t_1)}{\bar{a}(1 + t_1)(1 + t_2)(1 + t_3)} - \bar{a} + \frac{\bar{a}(1 + t_1)}{a(1 + t_1)(1 + t_2)(1 + t_3)} \tag{16}$$

or

$$t_1x - \bar{x} = at_1 - \frac{as_3(1 + t_1)}{\bar{a}\overset{3}{\pi}} - \bar{a} + \frac{\bar{a}(1 + t_1)}{a\overset{3}{\pi}}, \tag{17}$$

where $\overset{3}{\pi} = (1 + t_1)(1 + t_2)(1 + t_3)$; $s_3 = t_1t_2t_3$.

Let us replace t_1 in (17) by the parameter t, and we obtain the equation

$$tx - \bar{x} = at - \frac{as_3(1 + t)}{\bar{a}\overset{3}{\pi}} - \bar{a} + \frac{\bar{a}(1 + t)}{a\overset{3}{\pi}}. \tag{18}$$

Now then, if we let $t = t_2$, it is easily seen that (18) becomes the equation of a line on the reflexion of x_{31} and perpendicular to the line λ_2; similarly when $t = t_3$, (18) represents the line on the reflexion of x_{12} and perpendicular to the line λ_3.

Thus, by means of a process called "interpolation" we are enabled to write one equation, which as the parameter "t" assumes the values t_1, t_2, t_3 picks up the three lines μ_1, μ_2 and μ_3. Therefore, the three lines μ_1, μ_2 and μ_3 must intersect in a point, and their point of intersection is expressed by the equation

$$x_{123} = a - \frac{as_3}{\bar{a}\overset{3}{\pi}} + \frac{\bar{a}}{a\overset{3}{\pi}}. \tag{19}$$

Hence we have the theorem of Desboves:

Given three lines and an extra line; if we reflect the vertices of the three lines in the arbitrary line λ, *and then drop perpendiculars from these reflexions to the remaining line, the three perpendiculars will meet in a point.*

We will call this point the associated point of the line λ with reference to the three given lines λ_1, λ_2, λ_3.

1—*b*. Again, let us consider four undirected lines λ_1, λ_2, λ_3 and λ_4 as tangents to the parabola

$$x = \frac{1}{(1 + t)^2}. \tag{1}$$

We will recall that λ_i (for $i = 1, 2, 3, 4$) is expressed by the equation

$$x = \frac{1}{(1 + t_i)(1 + t)}. \tag{2}$$

For many purposes the circumcenter of a 3-line plays the part of the intersection of a 2-line. We will now find the circumcenter of three tangents to the parabola. First, let us examine the intersections of the 3-line formed by λ_1, λ_2, λ_3. On page 125 we defined the intersection of two tangents λ_i and λ_j as follows:

$$x_{ij} = \frac{1}{(1 + t_i)(1 + t_j)}. \tag{3}$$

Therefore, the intersections of λ_1, λ_2; λ_2, λ_3 and λ_3, λ_1 have as their respective equations

$$x_{12} = \frac{1}{(1 + t_1)(1 + t_2)}, \tag{4}$$

$$x_{23} = \frac{1}{(1 + t_2)(1 + t_3)}, \tag{5}$$

$$x_{31} = \frac{1}{(1 + t_3)(1 + t_1)}. \tag{6}$$

By means of interpolation we are enabled to write an equation that will pick up all three of these points, such an equation is

$$x = \frac{(1 + t)}{(1 + t_1)(1 + t_2)(1 + t_3)}, \tag{7}$$

for when $t = t_1$, (7) reduces to (5), and therefore (7) passes through the intersection of λ_2 and λ_3, that is, (7) is on the point x_{23}; when $t = t_2$, (7) reduces to (6), and is therefore on x_{31}; and finally when $t = t_3$, (7) reduces to (4) and passes through the point x_{12}. Consequently (7) represents a curve which passes through the three points x_{12}, x_{23} and x_{31}. (7) is the map-equation of a circle, whose center is

$$x_{123} = \frac{1}{(1 + t_1)(1 + t_2)(1 + t_3)}. \tag{8}$$

and the conjugate equation is

$$\bar{x}_{123} = \frac{s_3}{(1 + t_1)(1 + t_2)(1 + t_3)}. \tag{9}$$

This is then the circumcenter. Therefore, associated with four undirected lines, we will have four circumcenters, one for each 3-line.

We shall now proceed as we did in section 1—*a*, page 125; first we will reflect the circumcenter x_{123} in an arbitrary line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1 \tag{10}$$

and then we will erect a perpendicular to λ_4 on this reflexion. Since reflexion in a line is given by

$$\frac{x}{a}+\frac{\bar{y}}{\bar{a}}=1 \tag{11}$$

we will obtain as the reflexion of x_{123} in the line (10)

$$\frac{x}{a}=1-\frac{1}{\bar{a}}\left[\frac{s_3}{(1+t_1)(1+t_2)(1+t_3)}\right] \tag{12}$$

and its conjugate equation

$$\frac{\bar{x}}{\bar{a}}=1-\frac{1}{a}\left[\frac{1}{(1+t_1)(1+t_2)(1+t_3)}\right]. \tag{13}$$

Now since λ_4 is a tangent to the parabola

$$x=\frac{1}{(1+t)^2} \tag{14}$$

at the point t_4, we can write its equation as

$$x+\frac{\bar{x}}{t_4}=\frac{1}{1+t_4}. \tag{15}$$

A line $\perp$ to (15) will be of the form

$$x-\frac{\bar{x}}{t_4}=c-\frac{\bar{c}}{t_4}. \tag{16}$$

Therefore, the equation of a line perpendicular to λ_4, and on the reflexion of the point x_{123}, in the arbitrary line

$$\frac{x}{a}+\frac{\bar{x}}{\bar{a}}=1, \tag{17}$$

will have the following equation, namely,

$$x-\frac{\bar{x}}{t_4}=a-\frac{as'_3}{\bar{a}\overset{3}{\pi}}-\frac{\bar{a}}{t_4}+\frac{\bar{a}}{a\overset{3}{\pi}t_4} \tag{18}$$

or

$$t_4x-\bar{x}=at_4-\frac{as_4}{\bar{a}\overset{3}{\pi}}-\bar{a}+\frac{\bar{a}}{a\overset{3}{\pi}}, \tag{19}$$

where $\overset{3}{\pi}=(1+t_1)(1+t_2)(1+t_3)$; $s_4=t_1t_2t_3t_4$ and $s'_3=t_1t_2t_3$. Similarly, the equation of a line perpendicular to λ_3, and on the reflexion of the point x_{124}, in the arbitrary line, is

$$x-\frac{\bar{x}}{t_3}=a-\frac{as'_3}{\bar{a}\overset{3}{\pi}}-\frac{\bar{a}}{t_3}+\frac{\bar{a}}{a\overset{3}{\pi}t_3} \tag{20}$$

or

$$t_3 x - \bar{x} = a t_3 - \frac{a s_4}{a \overset{3}{\pi}} - \bar{a} + \frac{\bar{a}}{a \overset{3}{\pi}}, \tag{21}$$

where $\overset{3}{\pi} = (1 + t_1)(1 + t_2)(1 + t_4)$; $s_4 = t_1 t_2 t_3 t_4$ and $s_3' = t_1 t_2 t_4$.

Again, by the process called "interpolation," we are enabled to write an equation that will pick up these four lines, namely,

$$tx - \bar{x} = at - \frac{a s_4 (1 + t)}{\bar{a} \overset{4}{\pi}} - \bar{a} + \frac{\bar{a}(1 + t)}{a \overset{4}{\pi}} \tag{22}$$

(where $\overset{4}{\pi} = (1 + t_1)(1 + t_2)(1 + t_3)(1 + t_4)$; $s_4 = t_1 t_2 t_3 t_4$), for when $t = t_4$, (22) reduces to (19), and similarly when $t = t_3$, (22) reduces to (21), etc. Thus, it is easily seen that (22) picks up the four lines which are perpendicular to λ_1, λ_2, λ_3 and λ_4 respectively, and which also lie, respectively, on the reflexions of x_{234}, x_{134}, x_{124} and x_{123} in the line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1. \tag{23}$$

Moreover, (22) gives the intersection of these four perpendicular lines. Consequently, the point of intersection of the four lines is

$$x_{1234} = a - \frac{a s_4}{\bar{a} \overset{4}{\pi}} + \frac{\bar{a}}{a \overset{4}{\pi}} \tag{24}$$

and its conjugate equation is

$$\bar{x}_{1234} = \bar{a} - \frac{\bar{a}}{a \overset{4}{\pi}} + \frac{a s_4}{\bar{a} \overset{4}{\pi}},$$

where $\overset{4}{\pi} = (1 + t_1)(1 + t_2)(1 + t_3)(1 + t_4)$; $s_4 = t_1 t_2 t_3 t_4$.

Hence we can state the following theorem:

Given four lines and an extra line, if we reflect the circumcenters of the four 3-lines in the arbitrary line λ, and then drop perpendiculars from these reflexions to the remaining line, the four perpendiculars will meet in a point.

1—*c*. Now, we shall consider five undirected lines as the tangents of a double parabola. The equation of the double parabola is

$$x = \frac{A}{(\alpha - t)^2} + \frac{B}{(\beta - t)^2}, \tag{1}$$

provided the cusp condition $dx/dt = 0$; for t a turn. In this case the cusp condition is $\bar{A}\alpha^3 = A$; $\bar{B}\beta^3 = B$.

Let the five lines under consideration be designated as λ_1, λ_2, λ_3, λ_4 and λ_5. The equation of a tangent to the double parabola at the point t_1 is

$$x = \frac{A}{(\alpha - t_1)(\alpha - t)} + \frac{B}{(\beta - t_1)(\beta - t)} \tag{2}$$

or, in general, the equation of λ_i ($i = 1, 2, 3, 4, 5$) is

$$x = \frac{A}{(\alpha - t_i)(\alpha - t)} + \frac{B}{(\beta - t_i)(\beta - t)} . \tag{3}$$

We shall now proceed to find the intersection of two tangents, say λ_1 and λ_2. Since the equation of λ_1 is

$$x = \frac{A}{(\alpha - t_1)(\alpha - t)} + \frac{B}{(\beta - t_1)(\beta - t)} \tag{4}$$

and that of λ_2 is

$$x = \frac{A}{(\alpha - t_2)(\alpha - t)} + \frac{B}{(\beta - t_2)(\beta - t)} , \tag{5}$$

then the equation of x_{12}, the intersection of λ_1, λ_2, will be

$$x_{12} = \frac{A}{(\alpha - t_1)(\alpha - t_2)} + \frac{B}{(\beta - t_1)(\beta - t_2)} . \tag{6}$$

Similarly, we can find the intersections of x_{13}, x_{23}, x_{14}, x_{15}, etc.

In section 1—*b*, we found that to each 3-line there was associated a circumcenter; therefore to every 3-line arising from these five lines there is associated a circumcenter. Since we know the equation of the intersection of λ_1, λ_2, we can, by symmetry, write the equations of the intersections of λ_1, λ_3 and λ_2, λ_3. Thus

$$x_{12} = \frac{A}{(\alpha - t_1)(\alpha - t_2)} + \frac{B}{(\beta - t_1)(\beta - t_2)} , \tag{7}$$

$$x_{23} = \frac{A}{(\alpha - t_2)(\alpha - t_3)} + \frac{B}{(\beta - t_2)(\beta - t_3)} , \tag{8}$$

$$x_{31} = \frac{A}{(\alpha - t_3)(\alpha - t_1)} + \frac{B}{(\beta - t_3)(\beta - t_1)} . \tag{9}$$

By means of interpolation we can get the equation of the circumcircle, which passes through the points x_{12}, x_{23}, x_{31}, for if $t = t_i$ (where $i = 1, 2, 3$) in the equation

$$x = \frac{A(\alpha - t)}{(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)} + \frac{B(\beta - t)}{(\beta - t_1)(\beta - t_2)(\beta - t_3)} , \tag{10}$$

(10) becomes successively (8), (9) and (7).

The center of the circle (10) is

$$x_{123} = \frac{A\alpha}{(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)} + \frac{B\beta}{(\beta - t_1)(\beta - t_2)(\beta - t_3)} . \tag{11}$$

But from the four lines we can form four 3-lines, and since for every 3-line there is a circumcenter, therefore, from the four lines λ_1, λ_2, λ_3 and λ_4 we can obtain the four circumcenters x_{123}, x_{124}, x_{134}, x_{234}. From (11) we can write their equations, by means of symmetry.

Interpolating, we obtain

$$x = \frac{A\alpha(\alpha - t)}{(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)(\alpha - t_4)} + \frac{B\beta(\beta - t)}{(\beta - t_1)(\beta - t_2)(\beta - t_3)(\beta - t_4)}, \tag{12}$$

which equation, obviously, as t assumes successively the values t_1, t_2, t_3, t_4, picks up the points x_{123}, x_{124}, x_{134}, x_{234}.

But (12) is the equation of a circle, and its center is

$$x_{1234} = \frac{A\alpha^2}{(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)(\alpha - t_4)} + \frac{B\beta^2}{(\beta - t_1)(\beta - t_2)(\beta - t_3)(\beta - t_4)}, \tag{13}$$

$$\bar{x}_{1234} = \frac{\bar{A}(1/\alpha^2)}{(1/\alpha - 1/t_1)(1/\alpha - 1/t_2)(1/\alpha - 1/t_3)(1/\alpha - 1/t_4)} + \frac{\bar{B}(1/\beta^2)}{(1/\beta - 1/t_1)(1/\beta - 1/t_2)(1/\beta - 1/t_3)(1/\beta - 1/t_4)}. \tag{14}$$

Applying the cusp condition $\bar{A}\alpha^3 = A$; $\bar{B}\beta^3 = B$ and letting

$$P_4 = (\alpha - t_1)(\alpha - t_2)(\alpha - t_3)(\alpha - t_4)$$

and

$$Q_4 = (\beta - t_1)(\beta - t_2)(\beta - t_3)(\beta - t_4),$$

we have

$$\bar{x} = \frac{\bar{A}\alpha^2 s_4}{P_4} + \frac{\bar{B}\beta^2 s_4}{Q_4} \tag{15}$$

$$= \frac{A s_4}{\alpha P_4} + \frac{B s_4}{\beta Q_4} \tag{16}$$

$$= \sum^{2} \frac{A s_4}{\alpha P_4}, \tag{17}$$

where $s_4 = t_1 t_2 t_3 t_4$.

If now we reflect this point x_{1234} in the line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1, \tag{18}$$

we obtain, as its reflexion, the point

$$x = a - \frac{a}{\bar{a}} \sum^{2} \frac{As_4}{\alpha P_4}, \tag{19}$$

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2}{P_4}. \tag{20}$$

The equation of a tangent to the double parabola at the point t_5 is

$$x = \frac{A}{(\alpha - t_5)(\alpha - t)} + \frac{B}{(\beta - t_5)(\beta - t)} \tag{21}$$

and its conjugate equation is

$$\bar{x} = \frac{\bar{A}\alpha^2 tt_5}{(\alpha - t_5)(\alpha - t)} + \frac{\bar{B}\beta^2 tt_5}{(\beta - t_5)(\beta - t)}. \tag{22}$$

Since the cusp condition is $\bar{A}\alpha^3 = A$; $\bar{B}\beta^3 = B$, we can write (22) as

$$\frac{\bar{x}}{t_5} = \frac{At}{\alpha(\alpha - t_5)(\alpha - t)} + \frac{Bt}{\beta(\beta - t_5)(\beta - t)}. \tag{23}$$

Subtracting (23) from (21) we obtain a self-conjugate expression for the tangent to the double parabola at the point t_5, namely,

$$x - \frac{\bar{x}}{t_5} = \frac{A}{\alpha(\alpha - t_5)} + \frac{B}{\beta(\beta - t_5)} \tag{24}$$

or

$$x - \frac{\bar{x}}{t_5} = \sum^{2} \frac{A}{\alpha(\alpha - t_5)}. \tag{25}$$

A line perpendicular to this tangent and on the point $\bar{x}_{1234}$ is expressed by the equation

$$x + \frac{\bar{x}}{t_5} = a - \frac{a}{\bar{a}} \sum^{2} \frac{As_4}{\alpha P_4} + \frac{\bar{a}}{t_5} - \frac{\bar{a}}{at_5} \sum^{2} \frac{A\alpha^2}{P_4} \tag{26}$$

or

$$t_5 x + \bar{x} = at_5 - \frac{a}{\bar{a}} \sum^{2} \frac{As_5}{\alpha P_4} + \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2}{P_4}, \tag{27}$$

where

$$P_4 = (\alpha - t_1)(\alpha - t_2)(\alpha - t_3)(\alpha - t_4),$$
$$Q_4 = (\beta - t_1)(\beta - t_2)(\beta - t_3)(\beta - t_4)$$

and $S_5 = t_1t_2t_3t_4t_5$. But in this case we will have five perpendicular lines, namely, l_1 perpendicular to λ_1 and on $\bar{x}_{2345}$; $l_2 \perp \lambda_2$ and on $\bar{x}_{1345}$; $l_3 \perp \lambda_3$ and on $\bar{x}_{1245}$; $l_4 \perp \lambda_4$ and on $\bar{x}_{1235}$, and $l_5 \perp \lambda_5$ and on $\bar{x}_{1234}$.

Symmetrizing so as to pick up these five lines, we obtain the equation

$$tx + \bar{x} = at - \frac{a}{\bar{a}} \sum^{2} \frac{As_5(\alpha - t)}{\alpha P_5} + \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2(\alpha - t)}{P_5} \,. \tag{28}$$

It is at once evident, when $t = t_i$ (where $i = 1, 2, 3, 4, 5$), (28) gives us the line l_i ($i = 1, 2, 3, 4, 5$).

But (28) is the equation of the intersection of the five lines l_1, l_2, l_3, l_4 and l_5. This point has the equation

$$x_{12345} = a + \frac{a}{\bar{a}} \sum^{2} \frac{As_5}{P_5} + \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2}{P_5} \,. \tag{29}$$

Consequently, we have the theorem:

Given five lines and an extra line; if we reflect the centric points of the five 4-lines in an arbitrary line λ, and then drop perpendiculars from these reflexions to the remaining line, the five perpendiculars will meet in a point.

In general, we can say:

Given n lines and an extra line; if we reflect the centric points of the n "$(n - 1)$-lines" in an arbitrary line λ, and then drop perpendiculars from these reflexions to the remaining line, the n perpendiculars will meet in a point.

SECOND CHAIN OF THEOREMS.

2—*a*. Given four undirected lines λ_1, λ_2, λ_3 and λ_4 as tangents to the parabola

$$x = \frac{1}{(1 + t)^2} \tag{1}$$

at the points t_1, t_2, t_3 and t_4 respectively, then the equation of λ_i ($i = 1, 2, 3, 4$) is

$$x = \frac{1}{(1 + t_i)(1 + t)} \,. \tag{2}$$

With four undirected lines we can form four 3-lines for $C_3^4 = 4$. Here, the four 3-lines are composed of $\lambda_1, \lambda_2, \lambda_3$; $\lambda_1, \lambda_2, \lambda_4$; $\lambda_1, \lambda_3, \lambda_4$; and $\lambda_2, \lambda_3, \lambda_4$ respectively. Now, by the theorem of section 1—*a* we found that to every 3-line λ_i, λ_j, λ_k there is associated a point x_{ijk}. Consequently, to the 3-line λ_1, λ_2, λ_3 is associated a point x_{123}; to the 3-line λ_1, λ_2, λ_4 the point x_{124}, and so forth.

The equation of the point x_{123} is

$$x_{123} = a - \frac{as_3}{\bar{a}\overset{3}{\pi}} + \frac{\bar{a}}{a\overset{3}{\pi}} \,, \tag{3}$$

where $\overset{3}{\pi} = (1 + t_1)(1 + t_2)(1 + t_3)$; $s_3 = t_1 t_2 t_3$ and its conjugate equation is

$$\bar{x}_{123} = \bar{a} - \frac{\bar{a}}{a\overset{3}{\pi}} + \frac{as_3}{\bar{a}\overset{3}{\pi}}. \tag{4}$$

Similarly, the equation of the point x_{124} is

$$x_{124} = a - \frac{as_3}{\bar{a}\overset{3}{\pi}} + \frac{\bar{a}}{a\overset{3}{\pi}} \tag{5}$$

and

$$\bar{x}_{124} = \bar{a} - \frac{\bar{a}}{a\overset{3}{\pi}} + \frac{as_3}{\bar{a}\overset{3}{\pi}}, \tag{6}$$

where $\overset{3}{\pi} = (1 + t_1)(1 + t_2)(1 + t_4)$; $s_3 = t_1 t_2 t_4$. And so, in general, we have, as the equation of x_{ijk}, the point associated with the 3-line $\lambda_i, \lambda_j, \lambda_k$:

$$x_{ijk} = a - \frac{as_3}{\bar{a}\overset{3}{\pi}} + \frac{\bar{a}}{a\overset{3}{\pi}} \tag{7}$$

and

$$\bar{x}_{ijk} = \bar{a} - \frac{\bar{a}}{a\overset{3}{\pi}} + \frac{as_3}{\bar{a}\overset{3}{\pi}}, \tag{8}$$

where $\overset{3}{\pi} = (1 + t_i)(1 + t_j)(1 + t_k)$; $s_3 = t_i t_j t_k$.

Symmetrizing so as to pick up the four points x_{123}, x_{124}, x_{134}, x_{234}, we obtain the equation

$$x = a - \frac{as_4(1 + t)}{\bar{a}t\overset{4}{\pi}} + \frac{\bar{a}(1 + t)}{a\overset{4}{\pi}} \tag{9}$$

and

$$\bar{x} = \bar{a} - \frac{\bar{a}(1 + t)}{a\overset{4}{\pi}} + \frac{as_4(1 + t)}{\bar{a}t\overset{4}{\pi}}, \tag{10}$$

where $\overset{4}{\pi} = (1 + t_1)(1 + t_2)(1 + t_3)(1 + t_4)$; $s_4 = t_1 t_2 t_3 t_4$. Adding (9) and (10) we have

$$x + \bar{x} = a + \bar{a},$$

which is the equation of a vertical line.

Consequently, we have another theorem, namely:

Given four lines and an extra line λ, the four associated points, which arise from the four 3-lines and an arbitrary line λ, lie on a line.

Cwojdzinski states this theorem.*

2—*b*. Given five lines $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ as tangents to the double parabola

$$x = \sum^{2} \frac{A_i}{(\alpha_i - t)^2} \tag{1}$$

* *Archiv der Mathematik und Physik*, Vol. 1, 1901, p. 180.

(cusp condition being $\bar{A}_i\alpha_i^3 = A_i$), at the points t_1, t_2, t_3, t_4 and t_5 respectively, then the equation of the line λ_i ($i = 1, 2, 3, 4, 5$) is

$$x = \sum^{2} \frac{A_i}{(\alpha_i - t_i)(\alpha_i - t)}\,. \tag{2}$$

With five undirected lines we can form five 4-lines for $C_4^5 = 5$. Here the five 4-lines are composed of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$; $\lambda_1, \lambda_3, \lambda_4, \lambda_5$; $\lambda_1, \lambda_2, \lambda_4, \lambda_5$; $\lambda_1, \lambda_2, \lambda_3, \lambda_5$; $\lambda_2, \lambda_3, \lambda_4, \lambda_5$ respectively. Now, by theorem 1—*b* we found that to every 4-line $\lambda_i, \lambda_j, \lambda_k, \lambda_l$ there is associated a point x_{ijkl}. Consequently, to the above-named 4-lines are associated the points x_{1234}, x_{1345}, x_{1235}, x_{1245}, x_{2345}. The equation of an associated point given in section 1—*b* was for four lines taken as tangents to a parabola with the arbitrary line λ; here it will be necessary to find the equation of the associated point, the four lines being taken as tangents to a double parabola.

From 1—*c* we have the equation of the circumcenter for three lines $\lambda_1, \lambda_2, \lambda_3$ taken as tangents to a double parabola, namely,

$$x_{123} = \sum^{2} \frac{A\alpha}{\overset{3}{\pi}}\,, \tag{3}$$

and its conjugate is

$$\bar{x}_{123} = -\sum^{2} \frac{As_3}{\alpha\overset{3}{\pi}}\,. \tag{4}$$

Reflecting this point in the line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1 \tag{5}$$

we obtain the point

$$x = a + \frac{a}{\bar{a}}\sum^{2} \frac{As_3}{\alpha\overset{3}{\pi}}\,, \tag{6}$$

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a}\sum^{2} \frac{A\alpha}{\overset{3}{\pi}}\,. \tag{7}$$

A line $\perp$ to the tangent λ_4 and on the reflexion of the point x_{123} in the arbitrary line λ is

$$t_4x + \bar{x} = t_4a + \frac{a}{\bar{a}}\sum^{2} \frac{As_4}{\alpha\overset{3}{\pi}} + \bar{a} - \frac{\bar{a}}{a}\sum^{2} \frac{A\alpha}{\overset{3}{\pi}}\,. \tag{8}$$

Now interpolating so as to pick up these four perpendiculars, we obtain the equation

$$tx + \bar{x} = ta + \frac{a}{\bar{a}}\sum^{2} \frac{As_4(\alpha - t)}{\alpha\overset{4}{\pi}} + \bar{a} - \frac{\bar{a}}{a}\sum^{2} \frac{A\alpha(\alpha - t)}{\overset{4}{\pi}}\,, \tag{9}$$

which is an associated point

$$x = a - \frac{a}{\bar{a}} \sum^{2} \frac{As_4}{\alpha \overset{4}{\pi}} + \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha}{\overset{4}{\pi}}, \tag{10}$$

where $\overset{4}{\pi} = (\alpha_i - t_1)(\alpha_i - t_2)(\alpha_i - t_3)(\alpha_i - t_4)$; $s_4 = t_1t_2t_3t_4$.

Interpolating so as to pick up these five associated points, we have

$$x = a - \frac{a}{\bar{a}} \sum^{2} \frac{As_5(\alpha - t)}{\alpha t \overset{5}{\pi}} + \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha(\alpha - t)}{\overset{5}{\pi}}. \tag{11}$$

This is the equation of an ellipse, and its conjugate equation is

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2(\alpha - t)}{\overset{5}{\pi}} + \frac{a}{\bar{a}} \sum^{2} \frac{As_5(\alpha - t)}{t \overset{5}{\pi}}, \tag{12}$$

where $\overset{5}{\pi} = (\alpha_i - t_1)(\alpha_i - t_2)(\alpha_i - t_3)(\alpha_i - t_4)(\alpha_i - t_5)$; $s_5 = t_1t_2t_3t_4t_5$.

Hence we have the theorem:

Given five lines and an extra line λ, the five associated points, which arise from the five 4-lines, and an arbitrary line λ, lie on an ellipse.

2—*c*. Given six lines $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ as tangents to the double parabola

$$x = \sum^{2} \frac{A_i}{(\alpha_i - t)^2} \tag{1}$$

(cusp condition being $\bar{A}_i\alpha_i^3 = A$), at the points $t_1, t_2, t_3, t_4, t_5, t_6$ respectively, then the equation of the tangent λ_i $(i = 1, 2, 3, 4, 5, 6)$ is

$$x = \sum^{2} \frac{A_i}{(\alpha_i - t_i)(\alpha_i - t)}. \tag{2}$$

With six undirected lines we can form six 5-lines for $C_5^6 = 6$. From theorem 1—*c* we found that to every 5-line and an arbitrary line there is associated a point. Therefore to six lines and an arbitrary line there will be associated six points. The equation of the point x_{12345} is

$$x = a + \frac{a}{\bar{a}} \sum^{2} \frac{As_5}{\alpha \overset{5}{\pi}} + \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2}{\overset{5}{\pi}}. \tag{3}$$

Symmetrizing so as to pick up the six points, namely, x_{12345}, x_{12346}, x_{12456}, x_{13456}, x_{12356}, x_{23456}, we have

$$x = a + \frac{a}{\bar{a}} \sum^{2} \frac{As_6(\alpha - t)}{\alpha t \overset{6}{\pi}} + \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2(\alpha - t)}{\overset{6}{\pi}}, \tag{4}$$

which is the map-equation of an ellipse. Its conjugate equation is

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a}\sum^{2}\frac{A\alpha^3(\alpha - t)}{\underset{6}{\pi}} - \frac{a}{\bar{a}}\sum^{2}\frac{As_6(\alpha - t)}{t\underset{6}{\pi}}. \qquad (5)$$

Hence the theorem:

Given six lines and an extra line λ; the six associated points, which arise from the six 5-lines and an arbitrary line λ, lie on an ellipse.

2—*g*. In general, we can say:

Given n lines and an arbitrary line λ; the n associated points, which arise from the n "$(n - 1)$-lines" and an arbitrary line λ, lie on an ellipse.

Directed Lines.

First Chain of Theorems.

I—*a*. Suppose we have four directed lines L_1, L_2, L_3, L_4 given as tangents to a parastroid at the points t_1, t_2, t_3 and t_4; then since the equation of the parastroid is

$$t^4 - t^3x + \mu t^2 - t\bar{x} + 1 = 0 \qquad (1)$$

(where μ is real), the equations of L_1, L_2, L_3, L_4 will be respectively

$$t_1^4 - t_1^3x + \mu t_1^2 - t_1\bar{x} + 1 = 0, \qquad (2)$$
$$t_2^4 - t_2^3x + \mu t_2^2 - t_2\bar{x} + 1 = 0, \qquad (3)$$
$$t_3^4 - t_3^3x + \mu t_3^2 - t_3\bar{x} + 1 = 0, \qquad (4)$$
$$t_4^4 - t_4^3x + \mu t_4^2 - t_4\bar{x} + 1 = 0. \qquad (5)$$

The incenter* of the lines L_1, L_2, L_3 is

$$x_{123} = t_1 + t_2 + t_3 + \frac{1}{t_1t_2t_3} \qquad (6)$$

and its conjugate is

$$\bar{x}_{123} = 1/t_1 + 1/t_2 + 1/t_3 + t_1t_2t_3. \qquad (7)$$

The reflexion of the point x_{123} in the line λ, namely,

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1, \qquad (8)$$

is

$$x = a - \frac{a}{\bar{a}}[1/t_1 + 1/t_2 + 1/t_3 + t_1t_2t_3] \qquad (9)$$

and

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a}\left[t_1 + t_2 + t_3 + \frac{1}{t_1t_2t_3}\right]. \qquad (10)$$

We now desire to drop a perpendicular from the reflexion of x_{123} to the

* Hodgson, Joseph E., "Orthocentric Properties of the Plane Directed n-Line," *Transactions of the American Mathematical Society*, Vol. 13, 1912, p. 199.

line L_4; since a perpendicular to L_4 is of the form

$$t_4^2 x - \bar{x} = t_4^2 c - \bar{c}, \tag{11}$$

in order that this perpendicular pass through the reflexion of x_{123} in the line λ, (11) becomes

$$t_4^2 x - \bar{x} = t_4^2 \left[a - \frac{a}{\bar{a}} \left(\frac{s_2}{s_3} + s_3 \right) \right] - \left[\bar{a} - \frac{\bar{a}}{a} \left(s_1 + \frac{1}{s_3} \right) \right] \cdot \tag{12}$$

In order to interpolate, so as to be able to pick up the four perpendiculars, namely, the one from the reflexion of x_{234} to L_1; from the reflexion of x_{134} to L_2, etc., we will introduce the following symmetric functions:

$$\begin{aligned} s_1 &= t_1 + t_2 + t_3, & \sigma_1 &= t_1 + t_2 + t_3 + t_4, \\ s_2 &= t_1 t_2 + t_1 t_3 + t_2 t_3, & \sigma_2 &= t_1 t_2 + t_1 t_3 + t_1 t_4 + t_2 t_3 + t_2 t_4 + t_3 t_4, \\ s_3 &= t_1 t_2 t_3, & \sigma_3 &= t_1 t_2 t_3 + t_1 t_2 t_4 + t_1 t_3 t_4 + t_2 t_3 t_4, \\ & & \sigma_4 &= t_1 t_2 t_3 t_4; \end{aligned}$$

$$\begin{aligned} \therefore \quad s_1 &= \sigma_1 - t_4, \\ s_2 &= \sigma_2 - t_4 \sigma_1 - t_4^2, \\ s_3 &= \sigma_4 / t_4. \end{aligned}$$

Now then, making use of the above symmetric functions, (12) can be written as

$$t_4^2 x - \bar{x} = t_4^2 a - \frac{a}{\bar{a}} \left[\frac{(t_4^2 \sigma_2 - t_4^3 \sigma_1 + t_4^4) t_4}{\sigma_4} + \frac{t_4 \sigma_4^2}{\sigma_4} \right] - \left[\bar{a} - \frac{\bar{a}}{a} \left(s_1 + \frac{1}{s_3} \right) \right] \tag{13}$$

But

$$t^4 - \sigma_1 t^3 + \sigma_2 t^2 - \sigma_3 t + \sigma_4 = 0, \tag{14}$$

$$\therefore \quad t_4 - \sigma_1 t^3 + \sigma_2 t^2 = \sigma^3 t - \sigma_4. \tag{15}$$

Making use of this relation in (13), we obtain the equation

$$t_4^2 x - \bar{x} = t_4^2 a - \left[\frac{a t_4^2 \sigma_3 - a t_4 \sigma_4 + a t_4 \sigma_4^2}{\bar{a} \sigma_4} \right] - \left[\bar{a} - \frac{\bar{a}}{a} \left(s_1 + \frac{1}{s_3} \right) \right] \cdot \tag{16}$$

Thus (16) is the equation of a line which passes through the reflexion of x_{123}, and is perpendicular to L_4; which for convenience we will call R_4. There will be four such lines, namely, R_1, R_2, R_3, R_4, which will be obtained from the original four directed lines L_1, L_2, L_3 and L_4.

Symmetrizing so as to pick up these four lines, we have

$$t^2 x - \bar{x} = t^2 a - \left[\frac{a t^2 \sigma_3 - a t \sigma_4 + a t \sigma_4^2}{\bar{a} \sigma_4} \right] - \left[\bar{a} - \frac{\bar{a}}{a} \left(\sigma_1 - t + \frac{t}{\sigma_4} \right) \right], \tag{17}$$

which is a circle.

As a result, we have the theorem:

Given four directed lines and an extra line λ; if we reflect the incenters of the four 3-lines in an arbitrary line λ, and then drop perpendiculars from these reflexions to the remaining line, these four perpendiculars will touch a circle.

I—*b*. We will next consider the five directed lines L_1, L_2, L_3, L_4, L_5 as tangents to a parastroid, at the points t_1, t_2, t_3, t_4 and t_5 respectively. L_i (where $i = 1, 2, 3, 4, 5$) will be expressed by the equation

$$t_i^4 - t_i^3 x + \mu t_i^2 - t_i \bar{x} + 1 = 0, \tag{1}$$

where μ is real.

As given in section I—*a*, the incenter of the 3-line L_1, L_2, L_3 is

$$x_{123} = t_1 + t_2 + t_3 + \frac{1}{t_1 t_2 t_3}\,. \tag{2}$$

If now, instead of taking three directed lines, we take four directed lines, namely, L_1, L_2, L_3, L_4, then we will be able to form four 3-lines, and since there is an incenter associated with each 3-line, there will be four incenters associated with the four directed lines. Interpolating, so as to pick up these four incenters, we obtain the equation

$$x = \sigma_1 - t + \frac{t}{\sigma_4}, \tag{3}$$

where $\sigma_1 = t_1 + t_2 + t_3 + t_4$ and $\sigma_4 = t_1 t_2 t_3 t_4$. Now (3) is the equation of a circle whose center is

$$x_{1234} = \sigma_1. \tag{4}$$

We will now proceed to reflect this center in an arbitrary line, say,

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1. \tag{5}$$

Doing this, we obtain as the reflexion of x_{1234}

$$x = a - \frac{a}{\bar{a}}\left[\frac{\sigma_3}{\sigma_4}\right] \tag{6}$$

and its conjugate equation

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a}[\sigma_1]. \tag{7}$$

As in the previous cases, we desire to drop a perpendicular from the reflexion of x_{1234} to the remaining line L_5. The equation of such a perpendicular is

$$t_5^2 x - \bar{x} = t_5^2\left[a - \frac{a}{\bar{a}}\left(\frac{\sigma_3}{\sigma_4}\right)\right] - \left[\bar{a} - \frac{\bar{a}}{a}(\sigma_1)\right]\cdot \tag{8}$$

But in this case, since we have five directed lines, we will obtain five perpendiculars similar to the one expressed by equation (8). In order to interpolate we will have to introduce the following symmetric functions:

$$
\begin{aligned}
\sigma_1 &= t_1 + t_2 + t_3 + t_4, & P_1 &= \sum^{5} t_1,\\
\sigma_2 &= \sum^{6} t_1t_2, & P_2 &= \sum^{10} t_1t_2,\\
\sigma_3 &= \sum^{4} t_1t_2t_3, & P_3 &= \sum^{10} t_1t_2t_3,\\
\sigma_4 &= t_1t_2t_3t_4, & P_4 &= \sum^{5} t_1t_2t_3t_4,\\
& & P_5 &= t_1t_2t_3t_4t_5;
\end{aligned}
$$

$$
\begin{aligned}
\therefore\quad \sigma_1 &= P_1 - t_5,\\
\sigma_2 &= P_2 - P_1t_5 + t_5^2,\\
\sigma_3 &\doteq P_3 - t_5\sigma_2,\\
\sigma_4 &= P_5/t_5.
\end{aligned}
$$

But since the "Ps" can be considered as the symmetric functions of a quintic in t, we obtain the relation

$$t^5 - P_1t^4 + P_2t^3 - P_3t^2 + P_4t - P_5 = 0, \tag{9}$$

from which we have

$$t_5^2\sigma_3 = t_5^2P_3 - t_5^3\sigma_2 = t_5^2P_3 - t_5^3P_2 + t_5^4P_1 - t_5^5 \tag{10}$$

and also that

$$t_5^2P_3 - t_5^3P_2 + t_5^4P_1 - t_5^5 = t_5P_4 - P_5. \tag{11}$$

Making use of the relations that exist among these two sets of symmetric functions, (8) can be written as

$$t_5^2x - \bar{x} = t_5^2a - \frac{a}{\bar{a}}\left[\frac{(t_5^2P_3 - t_5^3\sigma_2)t_5}{P_5}\right] - \left[\bar{a} - \frac{\bar{a}}{a}(\sigma_1)\right] \tag{12}$$

or applying (11) to (12) we obtain

$$t_5^2x - \bar{x} = t_5^2a - \frac{a}{\bar{a}}\left[\frac{P_4t_5^2 - P_5t_5}{P_5}\right] - \left[\bar{a} - \frac{\bar{a}}{a}(\sigma_1)\right]\cdot \tag{13}$$

Interpolating so as to pick up the five perpendiculars, we have the equation

$$t^2x - \bar{x} = t^2a - \frac{a}{\bar{a}}\left[\frac{t^2P_4 - tP_5}{P_5}\right] - \left[\bar{a} - \frac{\bar{a}}{a}(P_1 - t)\right], \tag{14}$$

which is a circle.

Consequently, we have the following theorem:

Given five directed lines and an extra line; if we reflect the centric points of the five 4-lines in an arbitrary line λ, and then drop perpendiculars from these reflexions to the remaining line, these five perpendiculars will touch a circle.

I—*g*. In general, we can say:

Given n directed lines and an extra line; if we reflect the centric points of the n "$(n-1)$-lines" in an arbitrary line λ, and then drop perpendiculars from these reflexions to the remaining line, these n perpendiculars will touch a circle.

SECOND CHAIN OF THEOREMS.

II—*a*. Let us consider five directed lines L_1, L_2, L_3, L_4, L_5 as tangents to a parastroid, at the points t_1, t_2, t_3, t_4, t_5 respectively; then L_i (where $i = 1, 2, 3, 4, 5$) can be expressed as

$$t_i^4 - t_i^3 x + t_i^2 \mu - t_i \bar{x} + 1 = 0, \tag{1}$$

where μ is real.

By the previous theorem I—*a* we found that to every four directed lines and an extra line λ there is associated a circle, and since there are five directed lines, we will have five 4-lines and, therefore, five circles associated with the five directed lines. The equation of one of these circles is

$$t^2 x - \bar{x} = t^2 a - \left[\frac{at^2\sigma_3 - at\sigma_4 + at\sigma_4^2}{\bar{a}\sigma_4}\right] - \left[\bar{a} - \frac{\bar{a}}{a}\left(\sigma_1 - t + \frac{t}{\sigma_4}\right)\right]. \tag{2}$$

Symmetrizing so as to pick up the five circles, we obtain

$$\begin{aligned} t^2 x - \bar{x} = t^2 a - \left[\frac{at^2}{\bar{a}}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} + \frac{1}{t_5} - \frac{1}{T}\right)\right] + \frac{at}{\bar{a}} - \frac{at\sigma_5}{\bar{a}T} \\ - \bar{a} + \frac{\bar{a}}{a}\left[\sigma_1 - t - T + \frac{tT}{\sigma_5}\right], \end{aligned} \tag{3}$$

where

$$\sigma_1 = t_1 + t_2 + t_3 + t_4 + t_5;$$
$$\sigma_5 = t_1 t_2 t_3 t_4 t_5.$$

For convenience, let us denote σ_5 by T, then (3) becomes

$$\begin{aligned} t^2 x - \bar{x} = t^2 a - \left[\frac{at^2}{\bar{a}}\left(\frac{\sigma_4}{T} - \frac{1}{T}\right)\right] + \frac{at}{\bar{a}} - \frac{atT}{\bar{a}T} - \bar{a} \\ + \frac{\bar{a}}{a}\left[\sigma_1 - t - T - \frac{tT}{T}\right] \end{aligned} \tag{4}$$

or

$$t^2 x - \bar{x} = t^2 a - \left[\frac{at^2}{\bar{a}}\left(\frac{\sigma_4}{T} - \frac{1}{T}\right)\right] - \bar{a} + \frac{\bar{a}}{a}[\sigma_1 - T]. \tag{5}$$

Now then let $T = t$, and (5) becomes

$$t^2x - \bar{x} = t^2a - \left[\frac{at(\sigma_4 - 1)}{\bar{a}}\right] - \bar{a} + \frac{\bar{a}}{a}[\sigma_1 - t] \tag{6}$$

or

$$a\bar{a}(t^2x - \bar{x}) = a^2\bar{a}t^2 - a^2t(\sigma_4 - 1) - \bar{a}^2a + \bar{a}^2(\sigma_1 - t), \tag{7}$$

$$a\bar{a}(t^2x - \bar{x}) = a^2\bar{a}t^2 - a^2t\sigma_4 + a^2t - \bar{a}^2t + \bar{a}^2\sigma_1 - \bar{a}^2a, \tag{8}$$

$$a\bar{a}(t^2x - \bar{x}) = a^2\bar{a}t^2 - t(a^2\sigma_4 - a^2 + \bar{a}^2) + \bar{a}^2(\sigma_1 - a), \tag{9}$$

$$t^2x - \bar{x} = at^2 - \frac{t(a^2\sigma_4 - a^2 + \bar{a}^2)}{a\bar{a}} + \frac{\bar{a}(\sigma_1 - a)}{a}. \tag{10}$$

But $t = T = \sigma_5$,

$$\therefore\quad x - \frac{\bar{x}}{\sigma_5^2} = a - \frac{(a^2\sigma_4 - a^2 + \bar{a}^2)}{\sigma_5 a\bar{a}} + \frac{\bar{a}(\sigma_1 - a)}{\sigma_5^2 a}; \tag{11}$$

this is a self-conjugate equation, and is therefore the equation of a line, the line which touches the five circles arising from the five 4-lines.

Hence we have the theorem:

Given five directed lines and an extra line; the five circles, which arise from the five 4-lines and an arbitrary line λ, *all touch the same line.*

II—*b*. Let us consider six directed lines $L_1, L_2, L_3, L_4, L_5, L_6$ as tangents to a cyclogen, at the points $t_1, t_2, t_3, t_4, t_5, t_6$ respectively; then L_i $(i = 1, 2, 3, 4, 5, 6)$ can be expressed as

$$t_i^6 - ct_i^5 + xt_i^4 - \mu t_i^3 + \bar{x}t_i^2 - \bar{c}t_i + 1 = 0, \tag{1}$$

where $i = 1, 2, 3, 4, 5, 6$ and μ is real.

By a previous theorem I—*b* we found out that to every five directed lines and an arbitrary line λ there is associated a circle, and since there are six 5-lines which arise from six directed lines, there will be six circles associated with six directed lines.

The equation of one of these circles is

$$t^2x - \bar{x} = t^2a - \frac{a}{\bar{a}}\left[\frac{t^2P_4 - tP_5}{P_5}\right] - \left[\bar{a} - \frac{\bar{a}}{a}(P_1 - t)\right]. \tag{2}$$

Symmetrizing so as to pick up these six circles, we obtain the equation

$$\begin{aligned} t^2x - \bar{x} = t^2a - \frac{at^2}{\bar{a}}\left[\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} + \frac{1}{t_5} + \frac{1}{t_6} - \frac{1}{T}\right] + \frac{at}{\bar{a}} - \bar{a} \\ + \frac{\bar{a}}{a}[\sigma_1 - t - T], \end{aligned} \tag{3}$$

where $\sigma_1 = t_1 + t_2 + t_3 + t_4 + t_5 + t_6$; $\sigma_6 = t_1t_2t_3t_4t_5t_6$. For convenience,

let us denote σ_6 by T, then (3) becomes

$$t^2x - \bar{x} = t^2a - \frac{at^2}{\bar{a}}\left[\frac{\sigma_5}{T} - \frac{1}{T}\right] + \frac{at}{\bar{a}} - \bar{a} + \frac{\bar{a}\sigma_1}{a} - \frac{\bar{a}t}{a} - \frac{\bar{a}T}{a}, \tag{4}$$

$$t^2x - \bar{x} = t^2a - \frac{at^2}{\bar{a}}\left[\frac{\sigma_5}{T} - \frac{1}{T}\right] + \frac{a^2t}{a\bar{a}} - \bar{a} + \frac{\bar{a}\sigma_1}{a} - \frac{\bar{a}^2t}{a\bar{a}} - \frac{\bar{a}T}{a}. \tag{5}$$

Now then, let $T = t$, and (5) becomes

$$t^2x - \bar{x} = t^2a - \left[\frac{at}{\bar{a}}(\sigma_5 - 1)\right] + \frac{a^2t}{a\bar{a}} - \frac{\bar{a}^2t}{a\bar{a}} - \bar{a} + \frac{\bar{a}}{a}(\sigma_1 - t), \tag{6}$$

$$a\bar{a}(t^2x - \bar{x}) = t^2a^2\bar{a} - a^2t(\sigma_5 - 1) + a^2t - \bar{a}^2t - a\bar{a}^2 + \bar{a}^2(\sigma_1 - 1), \tag{7}$$

$$a\bar{a}(t^2x - \bar{x}) = a^2\bar{a}t^2 + t(2a^2 - 2\bar{a}^2 - a^2\sigma_5) + \bar{a}^2(\sigma_1 - a), \tag{3}$$

$$t^2x - \bar{x} = at^2 + \frac{t(2a^2 - 2\bar{a}^2 - a^2\sigma_5)}{a\bar{a}} + \frac{\bar{a}(\sigma_1 - a)}{a}. \tag{9}$$

But $t = T = \sigma_6$,

$$\therefore\ x - \frac{\bar{x}}{\sigma_6^2} = a + \frac{(2a^2 - 2\bar{a}^2 - a^2\sigma_5)}{\sigma_6 a\bar{a}} + \frac{\bar{a}(\sigma_1 - a)}{\sigma_6^2 a}; \tag{10}$$

this is a self-conjugate equation, and is therefore the equation of a line, the line which touches the six circles arising from the six 5-lines.

Consequently we have the theorem:

Given six directed lines and an extra line; the six circles, which arise from the six 5-lines and an arbitrary line λ, all touch the same line.

And, in general, we can say:

II—c. Given n directed lines and an extra line; the n circles, which arise from the n "(n − 1)-lines" and an arbitrary line λ, all touch the same line.

A PORISTIC SYSTEM OF EQUATIONS.

By L. B. Robinson.

In an attempt to reduce a certain system of partial differential equations to a canonical form, the author has met with some systems of poristic equations of which the following system is a simple type:

$$P_{rs} \equiv \sum_{k=0}^{n} \sum_{i=0}^{n} p_{ki} x_{kr} x_{is} = 0 \quad (r, s = 1, 2, \cdots, n). \tag{1}$$

$p_{ki} = p_{ik}$, $P_{rs} = P_{sr}$, the matrix $\| x_{kr} \|$ is of rank n.

These equations imply that n points on the quadric spread

$$Q \equiv \sum_{k=0}^{n} \sum_{i=0}^{n} p_{ki} x_k x_i = 0$$

are conjugate in pairs. This is true only when the quadric spread consists of two hyperplanes; consequently we infer that the equations (1) form a poristic set possessing no relevant solution at all unless certain conditions are satisfied and an infinite number of solutions when the conditions are satisfied. It is clear, in fact, that when the quadric spread does consist of two hyperplanes a set of n points on one hyperplane satisfies the requirements.

The case $n = 2$ is discussed by Clebsch,* and his method can be extended so as to cover the general case. It may be worth while to indicate the necessary analysis, as it can be extended to the case of a form of the fourth order; also because the conditions are, as M. Janet has pointed out, those under which the differential equation

$$\sum_{k=0}^{n} \sum_{i=0}^{n} p_{ki} \frac{\partial^2 u}{\partial x_k \partial x_i} = 0$$

can be reduced to the form

$$\frac{\partial}{\partial \xi_0} \sum_{m=1}^{n} \alpha_m \frac{\partial u}{\partial \xi_m} = 0.$$

Let us denote the determinant $|x_{ij}|$ $(i, j = 0, 1, 2, \cdots, n)$ by the symbol D, the minor of x_{00} in this determinant by the symbol Δ_{00}, and the minor of x_{ij} in the determinant Δ_{00} by the symbol M_{ij} $(i, j = 1, 2, \cdots, n)$. We shall also write

$$\sum_{s=1}^{n} x_{0s} M_{is} \equiv \Delta_{i0}.$$

* "Vorlesungen über Geometrie," Bd. I (new edition), p. 136.

We now proceed to reduce the system of equations (1) to a canonical form. With this end in view, we must write the sum

$$S^{(iktl)} \equiv 2\sum_{r=1}^{n} x_{kr}x_{ir}M_{tr}M_{lr} + \sum_{r=1}^{n}\sum_{s=1}^{n}(x_{kr}x_{is} + x_{ks}x_{ir}) \quad \begin{pmatrix} k, l, t = 1, 2, \cdots, n \\ i = 0, 1, 2, \cdots, n \end{pmatrix},$$

where $r > s$ in the double summation, and $i < k$, $t \gtrless l$ throughout. The above sum falls into two parts

$$\begin{aligned} S_1^{(iktl)} &\equiv \sum_{r=1}^{n} x_{kr}x_{ir}M_{tr}M_{lr} + \sum_{r=1}^{n}\sum_{s=1}^{n}[x_{kr}x_{is}M_{tr}M_{ls} + x_{ks}x_{ir}M_{ts}M_{lr}] \\ &\equiv \sum_{r=1}^{n} x_{kr}M_{tr}\sum_{s=1}^{n} x_{is}M_{ls} \end{aligned}$$

and $S_2^{(iktl)}$ obtained from $S_1^{(iktl)}$ by interchanging t and l. If $i \neq l$ and $i \neq 0$, the sum $\sum_{s=1}^{n} x_{is}M_{ls}$ must vanish, because it is then equal to a determinant two of whose columns are identical. Also if $k \neq t$ and $k \neq 0$ the sum $\sum_{r=1}^{n} x_{kr}M_{tr}$ vanishes for the same reason. Since $i < k$ we must assume that k is different from 0.

If we assume that $i = 0$, $k \neq 0$,

$$S_1 \equiv \sum_{r=1}^{n} x_{kr}M_{tr}\sum_{s=1}^{n} x_{0s}M_{ls}.$$

Furthermore if $k \neq t$, S_1 vanishes identically. Therefore we shall consider only the expression

$$S_1 \equiv \sum_{r=1}^{n} x_{tr}M_{tr}\sum_{s=1}^{n} x_{0,\,s}M_{ls},$$

where $k = t$. But

$$\sum_{r=1}^{n} x_{tr}M_{tr} \equiv \Delta_{00}, \qquad \sum_{s=1}^{n} x_{0,\,s}M_{ls} \equiv \Delta_{l0},$$

$$\therefore \quad S_1 \equiv \Delta_{l0}\Delta_{00}.$$

We now consider S_2. Interchange the rôle of t and l, and clearly it reduces to $S_2 \equiv \Delta_{t0}\Delta_{00}$.

One case only remains for consideration. Let $i = t$, $k = l$. Then $S_2 = \Delta_{00}^2$, while $S_1 = 0$. We cannot interchange t and l in this case for $i < k$ and $t \leqq l$.

We have assumed while discussing S that $i < k$. But we shall also be obliged to consider the set $S^{(iitl)}$ obtained from S by dividing it by 2 and assuming that $i = k$. The result is equal to the product

$$\sum_{r=1}^{n} x_{ir}M_{tr}\sum_{s=1}^{n} x_{is}M_{ls},$$

whose first factor vanishes unless $i = t$ or $i = 0$. In the first case, if $i > l$, the second factor reduces to a determinant, two of whose columns are equal; therefore the product vanishes. If $i = l$, the product becomes

$$\sum_{r=1}^{n} x_{ir}M_{tr}\sum_{s=1}^{n} x_{is}M_{ls} \equiv \Delta_{00}^2.$$

In the second case, the product becomes

$$\sum_{r=1}^{n} x_{0r}M_{tr}\sum_{s=1}^{n} x_{0s}M_{ls} \equiv \Delta_{l0}\Delta_{t0}.$$

In other words $\frac{1}{2}S^{(00tl)} \equiv \Delta_{l0}\Delta_{t0}$.

We next construct the sum

$$U \equiv \sum_{r=1}^{n} M_{tr}M_{lr}P_{rr} + \sum_{r=1}^{n}\sum_{s=1}^{n} (M_{tr}M_{ls} + M_{ts}M_{lr})P_{rs},$$

where $r > s$ in the double summation. From what has been previously written we see that

$$U \equiv \sum_{j=1}^{n} p_{jj}\frac{S^{(jjtl)}}{2} + p_{00}\frac{S^{(00tl)}}{2} + \sum_{i=1}^{n}\sum_{k=1}^{n} p_{ki}S^{(iktl)} + \sum_{k=1}^{n} p_{k0}S^{(0ktl)}.$$

We assume first that $t \neq l$. Then, as we have seen, $\frac{1}{2}S^{(jjtl)}$ vanishes, unless $j = 0$, when it is equivalent to Δ_{00}^2. As for the third member of U, its elements vanish unless $i = t$, $k = l$, when we obtain

$$p_{lt}S^{(tltl)} \equiv \Delta_{00}^2 p_{lt}.$$

The fourth member vanishes unless $k = l$ or $k = t$. So we can write down

(2) $\quad U \equiv \Delta_{00}^2 p_{lt} + \Delta_{t0}\Delta_{l0}p_{00} + \Delta_{l0}\Delta_{00}p_{t0} + \Delta_{t0}\Delta_{00}p_{l0} \quad (l, t = 1, 2, \cdots, n).$

When $t = l$, we have seen that all the terms of the first summation vanish save

$$p_{ll}\frac{S^{(llll)}}{2} = \Delta_{00}p_{ll}.$$

All the terms of the third summation vanish unless $i = k = l$; but this cannot happen since $i < k$. We see readily that (2) becomes

(2′) $\quad U \equiv \Delta_{00}^2 p_{ll} + \Delta_{l0}^2 p_{00} + 2\Delta_{l,0}\Delta_{00}p_{l,0}.$

The set formed by the union of (2) and (2′) may be called the canonical form of system (1), because what solves one system clearly solves the other. The relation known as "correlation multiplicatoire"* exists between (2) and (2′).

* Riquier, "Les systèmes d'équations aux dérivées partielles," p. 255.

Write the polynomial

$$(3)\quad (p_{12}p_{13} - p_{12}p_{23})(p_{12}p_{10} - p_{11}p_{20}) - (p_{12}^2 - p_{12}p_{22})(p_{13}p_{10} - p_{11}p_{30}).$$

We have demonstrated that the vanishing of the expressions

$$(A)\quad \Delta_{00}^2 p_{ii} + \Delta_{i0}^2 p_{00} + 2\Delta_{i0}\Delta_{00}p_{i0} + \Delta_{00}^2 p_{ij} + \Delta_{i0}\Delta_{j0}p_{00} \quad (i, j = 1, 2, 3).$$

is a necessary consequence of the vanishing of the system of polynomials (1). Let us now assume that $\Delta_{00} \neq 0$. Then we eliminate p_{11}, p_{12}, p_{13}, p_{22}, p_{23} from the polynomial (3) with the aid of the six expressions (A), and since (3) vanishes, we conclude that

The vanishing of the system of polynomials (1) *carries with it as a necessary consequence the vanishing of the polynomial* (3) *unless* $\Delta_{00} = 0$.

We have demonstrated that if we multiply each of the polynomials of the system (1) by certain multipliers and then sum the results we obtain the expressions (2) and (2′). This is equivalent to solving the set (1) with respect to the quantities p_{ij} where $i, j = 1, 2, \cdots, n;\ i \gtreqless j$.

Let us consider the effect of interchanging any two columns of the matrix

$$W \equiv \|x_{ij}\| \quad (i = 0, 1, 2, \cdots, n;\ j = 1, 2, \cdots, n).$$

By such a permutation, the system of polynomials (1) is unaltered. The polynomials (2) will be changed, if we operate on them with all possible permutations of the above-described type, into a set of sets of polynomials which we shall denote by the symbols (2_1), (2_2), $\cdots$. This change however is only with regard to outward form. For the above permutations simply enable us to solve the system (1) with respect to another set of coefficients. For example, if we interchange the first and second columns of the matrix W we shall solve (1) with respect to the quantities p_{ij} where $i, j = 0, 2, 3, \cdots, n;\ j \geqq i$. It is now evident that (2), (2_1), (2_2), $\cdots$ are numerically equivalent.

On the other hand, the polynomial (3) is changed by the permutations of the type above described into a chain of other polynomials $(n + 1)!$ in number. For example, the permutation (2, 4) transforms (3) into the polynomial

$$(p_{14}p_{13} - p_{11}p_{34})(p_{14}p_{10} - p_{11}p_{40}) - (p_{14}^2 - p_{11}p_{44})(p_{13}p_{10} - p_{11}p_{30}).$$

Let us designate this chain of polynomials by the symbol (3′). We can now announce the following theorem:

If $\Delta_{00} \neq 0$, *the vanishing of the system of polynomials* (1) *is a sufficient condition for the vanishing of the polynomial* (3) *and also of the chain of polynomials* (3′).

Consider the case $\Delta_{00} = 0$. Assume for the moment that $\Delta_{n0} \neq 0$ for

$n \neq 0$. Interchange the first and the nth column of the matrix W. As remarked above, such interchange simply replaces the system (1) by itself and transforms the system (2) into a system numerically equivalent. In other words instead of solving (1) with respect to p_{ij} where $i, j = 1, 2, \cdots, n$, we solve (1) with respect to p_{ij} where $i, j = 0, 1, 2, \cdots, n - 1$. In general we may assert

If one of the determinants $\Delta_{10}, \Delta_{20}, \cdots, \Delta_{00}$ does not vanish, the system of polynomials (1) *cannot vanish unless all the polynomials of system* (A) *vanish.*

We now consider the polynomial

$$(p_{12}^2 - p_{11}p_{22})(p_{13}^2 - p_{11}p_{33}) - (p_{12}p_{13} - p_{11}p_{23})^2. \tag{4}$$

If we equate the polynomials (A) to zero we see at once that the polynomial (4) must also be equal to zero.

Operate on this polynomial with the permutations we have previously used and we shall obtain a chain of polynomials which we shall denote by the symbol (4′). Following the line of reasoning employed in the case of the system of polynomials (4) we demonstrate the following theorem:

If any one of the determinants $\Delta_{i0} \neq 0$ $(i = 0, 1, 2, \cdots, n)$, then the vanishing of the system of polynomials (1) *carries with it the vanishing of all the polynomials of system* (3′).

We have now derived two sets of conditions (3) and (3′) which must be satisfied if the system of polynomials (1) has a solution which does not cause all the determinants of matrix W to vanish. These conditions are also sufficient as we shall proceed to show.

Write

$$p_{1i}^2 - p_{11}p_{ii} \equiv w_{ii}, \qquad p_{1i}p_{1j} - p_{11}p_{ij} \equiv w_{ij} \equiv w_{ji} \quad (i, j = 0, 1, 2, \cdots, n).$$

When we equate to zero all the minors of the second order in the determinant $|w_{ij}|$, we obtain a set of necessary conditions. As we shall show, these conditions are sufficient, indeed more than sufficient, but they are retained because of certain invariant properties to be considered in a subsequent paper. We shall demonstrate the sufficiency of these conditions for $n = 3$.*

Consider first the equations belonging to set (2′), that is,

$$\Delta_{00}^2 p_{ii} + 2\Delta_{i0}\Delta_{00}p_{i0} + \Delta_{i0}^2 p_{00} = 0 \quad (i = 1, 2, 3).$$

Write $\Delta_{i0}/\Delta \equiv \lambda_{i0}$. Solving the above equations, we get

$$\lambda_{i0} = \frac{-p_{i0} \pm \sqrt{v_{ii}}}{p_{00}}, \qquad v_{ji} \equiv v_{ij} \equiv p_{0i}p_{0j} - p_{ij}p_{00}.$$

* If $n = 3$ we can construct an equation with the w_{ij} for coefficients which, when written in lines, has coefficients whose vanishing gives the necessary and sufficient conditions for our theorem.

Suppose all the radicals chosen with the positive sign and substitute the above values of λ_{i0} in the three equations of the above set. The results are

$$(5) \qquad v_{12} = \sqrt{v_{11}v_{22}}, \qquad v_{13} = \sqrt{v_{11}v_{33}}, \qquad v_{23} = \sqrt{v_{22}v_{33}}.$$

If all the radicals are chosen with the negative sign the results are essentially the same. We see at once that from the conditions (5) follow the conditions

$$v_{23}v_{13} - v_{12}v_{33} = 0, \qquad v_{23}v_{12} - v_{13}v_{22} = 0, \qquad v_{13}v_{12} - v_{11}v_{23} = 0,^*$$

regardless of the choice of the sign of the radicals.

All other conditions must depend on the six written down above. These six can be written in determinant form.

The same reasoning applies when n is arbitrary.

The polynomials (3) and (3′) are not all independent, and they do not constitute a set of invariants under any linear transformation of coördinates although an invariant property is characterized by the vanishing of all of them simultaneously. They belong to the type of functions which Riquier has utilized to form the conditions that systems of partial differential equations may be passive. If I_s denotes a set of such functions they are transformed into a set I'_r connected with the I_s by relations of the type $I'_r = \sum f_{rs}I_s$ in which the determinant $|f_{rs}|$ is a power of the Jacobian of the transformation.†

If we wish to discover under what conditions a quadratic form in any number of variables can be expressed as the sum of two degenerate quadratic forms we write down the two matrices

$$W_1 \equiv \|x_{ij}\|, \quad i = 1, 2, \cdots, n-1, \qquad j = 1, \cdots, n-1, n,$$
$$W_2 \equiv \|x_{ij}\|, \quad i = 1, 2, \cdots, n-1, \qquad j = 0, 1, \cdots, n-1.$$

The expressions $\Delta_{t,\,0}$ are formed from the matrix $\overline{W}$ in the manner we described at the beginning of this article.

The expressions Δ_{tn} are formed from the matrix W in exactly the same manner.‡ Starting from the system of equations

$$P_{rs} = 0 \quad (r = 1, 2, \cdots, n-1; \quad s = 0, 1, 2, \cdots, n-1, n)$$

we can obtain the canonical form $\Delta^2_{00}p_{lt} + \Delta_{t,\,0}\Delta_{l,\,0}p_{00} + \Delta_{l,\,0}\Delta_{00}p_{t0}$

* These conditions are retained because of certain invariant conditions to be discussed in the next paper.

† See Riquier, "Les systèmes d'équations aux dérivées partielles." Riquier, being chiefly interested in establishing the existence of solutions of the most general type, has, as far as the author knows, said nothing about the invariant properties of his passivity conditions. But we can immediately verify by a few simple examples that they belong to the type of invariant conditions just defined.

‡ Note that $\Delta_{nn} \equiv \Delta_{00}$.

$+ \Delta_{t0}\Delta_{00}p_{l,0} + (\Delta_{tn}\Delta_{l0} + \Delta_{t0}\Delta_{ln})p_{n0} + \Delta_{tn}\Delta_{ln}p_{nn} + \Delta_{ln}\Delta_{nn}p_{tn} + \Delta_{tn}\Delta_{nn}p_{ln}$ $(l, t = 1, 2, \cdots, n-1)$.

When the above system is compatible, it is possible to express the quadratic form as the sum of two degenerate quadratics. The compatibility conditions could be calculated in a given numerical case, but such a calculation is exceeding long. However, we can enunciate the following theorem:

Given the polynomials $P_{rs} = 0$ $(s = 1, 2, \cdots, \lambda, \cdots, \omega;\ r = 1, 2, \cdots, \lambda)$. *The canonical form is a function of* $\lambda(\omega - \lambda)$ *determinants of the matrix* $\|x_{rs}\|$. *The system* $P_{rs} = 0$ *involves* $\lambda\omega$ *variables.* Thus the calculation of the resultant is shortened.

We shall now proceed to treat the case of a quartic form by a similar process. The general quartic may be written symbolically as follows:

$$P_{rrrr} \equiv (a_1x_{1r} + a_2x_{2r} + \cdots + a_nx_{nr} + a_0x_{0r})^4$$

where $a_1^4 = a_{1111}$, $a_1^3a_2 = a_{1112}$, etc. Form the first, second and third polars of each of the above expressions with respect to each of the points A_s whose coördinates are $(x_{1s}, x_{2s}, \cdots, x_{0s})$ respectively, s having the values $1, 2, \cdots, n$.*

If we desire to find the conditions under which the n-ary quartic is a product of a linear factor and a cubic factor, it is necessary to write down in parametric form the equations of a hyperplane. Let us therefore write

$$\rho x_{i1} = y_{i1} + \sum_{k=2}^{n} \lambda^{(k-1)} y_{ik} \quad (i = 1, 2, \cdots, n, 0)$$

and substitute these values for the x's in the polynomials P_{1111}. Equating to zero the coefficients of the λ's, we obtain a system of equations similar to $P_{rrrr} = 0$ and the equations derived from it by the polar process, except that we have the y's in place of the x's.

The argument applied by Clebsch to the conic extends itself immediately to the n-ary quartic. If the system of equations derived in the way just described can be solved in such a way that the solution does not cause all the determinants of the matrix

$$\|y_{ij}\| \quad (i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, n, 0)$$

to vanish, the quartic can be decomposed into a linear and a cubic factor. In the following paragraph when we write $P_{i\cdot\alpha,\, j\cdot\beta,\, k\cdot\gamma,\, l\cdot\delta}$ and $a_{i\cdot\alpha,\, j\cdot\beta,\, k\cdot\gamma,\, l\cdot\delta}$ it means that the subscript α occurs i times, the subscript β, j times, etc.

If we repeat the reasoning applied to a form of the second order we can establish the following identity.

* Denote this system by the symbol (1′).

$$\sum \frac{4!}{i!j!k!l!} M^i_{t\alpha} M^j_{t\beta} M^k_{t\gamma} M^l_{t\delta} P_{i\cdot\alpha,\, j\cdot\beta,\, k\cdot\gamma,\, l\cdot\delta}$$
$$\equiv \Delta^4 a_{4\cdot t} + 4\Delta^3 \Delta_t a_{3\cdot t,\,0} + 6\Delta^2 \Delta_t^2 a_{2\cdot t,\,2\cdot 0} + 4\Delta\Delta_t^3 a_{t,\,3\cdot 0} + \Delta_t^4 a_{4\cdot 0}$$
$$(\alpha, \beta, \gamma, \delta, t = 1, 2, \cdots, n;\ i + j + k + l = 4).$$

The summation extends over all possible integral values of the letters involved. Similarly we find that

$$\frac{1}{2\cdot 3\cdot 4}\sum_{i=1}^{n} M_{qi}\frac{\partial}{\partial M_{ti}} \cdot \sum_{i=1}^{n} M_{si}\frac{\partial}{\partial M_{ti}}$$
$$\times \sum_{i=1}^{n} M_{ri}\frac{\partial}{\partial M_{ti}} \sum \left[\frac{4!}{i!j!k!l!} M^i_{t\alpha} M^j_{t\beta} M^k_{t\gamma} M^l_{t\delta} P_{i\cdot\alpha,\, j\cdot\beta,\, k\cdot\gamma,\, l\cdot\delta}\right]$$

$$\begin{aligned}(6)\quad &\equiv \Delta^4 a_{tqsr} + \Delta^3\{\Delta_t a_{qsr0} + \Delta_q a_{tsr0} + \Delta_s a_{tqr0} + \Delta_r a_{tqs0}\}\\ &+ \Delta^2\{\Delta_s\Delta_r a_{tq,\,2\cdot 0} + \Delta_q\Delta_r a_{ts,\,2\cdot 0} + \Delta_q\Delta_s a_{tr,\,2\cdot 0} + \Delta_t\Delta_r a_{qs,\,2\cdot 0}\\ &+ \Delta_t\Delta_s a_{qr,\,2\cdot 0} + \Delta_t\Delta_q a_{sr,\,2\cdot 0}\} + \Delta\{\Delta_q\Delta_s\Delta_r a_{t,\,3\cdot 0} + \Delta_t\Delta_q\Delta_r a_{s,\,3\cdot 0}\\ &+ \Delta_t\Delta_q\Delta_s a_{r,\,3\cdot 0} + \Delta_t\Delta_s\Delta_r a_{q,\,3\cdot 0}\} + \Delta_t\Delta_q\Delta_r\Delta_s a_{4\cdot 0}.\end{aligned}$$

To compute the value of the expression

$$\frac{1}{3\cdot 4}\sum_{i=1}^{n} M_{si}\frac{\partial}{\partial M_{ti}} \sum_{i=1}^{n} M_{ri}\frac{\partial}{\partial M_{ti}} \sum \frac{4!}{i!j!k!l!}[M^i_{t\alpha} M^j_{t\beta} M^k_{t\gamma} M^l_{t\delta} P_{i\cdot\alpha,\, j\cdot\beta,\, k\cdot\gamma\, l\cdot\delta}]$$

let $q = r$ in the right-hand member of the equation (6). To compute the value of the expression

$$\frac{1}{4}\sum_{i=1}^{n} M_{ri}\frac{\partial}{\partial M_{ti}} \sum \frac{4!}{i!j!k!l!} M^i_{t\alpha} M^j_{t\beta} M^k_{t\gamma} M^l_{t\delta} P_{i\cdot\alpha,\, j\cdot\beta,\, k\cdot\gamma,\, l\cdot\delta},$$

in addition to setting $q = r$, we must set $s = t$ in the right-hand member of equation (6).

Equate the right-hand members of all the equations just obtained to zero and denote the resulting system of equations by (A′). These represent a set of conditions which must necessarily be satisfied if the quartic form contains a linear factor. (A′) and (1′) are in "correlation multiplicatoire," for, even as such a relation has been shown to exist in the case of the quadric form, it exists in the case in question. Therefore if all the equations (A′) are satisfied and no inconsistency is introduced, we have the conditions, both necessary and sufficient. To find whether all the equations are consistent, we proceed as before, except that we now have to solve a set of equations of the fourth degree

$$\Delta^4 a_{4t} + 4\Delta^3\Delta_t a_{3\cdot t,\,0} + 6\Delta^2\Delta_t^2 a_{2\cdot t,\,2\cdot 0} + 4\Delta\Delta_t^3 a_{t,\,3\cdot 0} + \Delta_t^4 a_{4\cdot 0} = 0$$

to obtain the ratio Δ_t/Δ $(t = 1, 2, \cdots, n)$.

Substituting the resulting values of Δ_i/Δ in the remaining equations of the system (A′) we obtain a set of equations involving the a's alone. If these are satisfied, it is possible to find an infinite number of sets of solutions of our equations which do not cause to vanish all the determinants of the matrix

$$\|x_{ij}\| \qquad (i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, n, 0).$$

We have thus reduced the problem of finding the conditions under which a quartic contains a linear factor to the solution of a binary quartic. Had we attacked the problem directly, i.e., had we written

$$(\alpha x)^4 = (\beta x)(\gamma x)^3,$$

we could have obtained the conditions, but their exact form would have varied according to the order of calculation which we should elect to follow. Proceeding according to one particular order, the author obtained a system of equations similar in some respects to (A′), but less simple. For example in the ternary case the equation corresponding to the first equation of set (A′) is

$$81\Delta^4 - 324\Delta^3 a_{1112} + (432a_{1112}^2 - 54a_{1122})\Delta^2 - (192a_{1112}^3 + 12a_{1222})\Delta \\ - (16a_{1112}a_{1222} - 96a_{1112}^2 a_{1122} - a_{2222}) = 0.$$

This equation has been simplified by setting $a_{1111} = 1$.

As the solution of a quartic equation is generally a long expression, it is better to proceed as follows in a numerical case.

Eliminating Δ_{10} from the equation of the fourth degree in Δ_{10}/Δ and the equation which is linear in Δ_{10} and of the third degree in Δ_{20}, we obtain an equation of the fourth degree in Δ_{20}. Combining this with the equation for Δ_{20} in the set (A′), we may find a common root of the two equations by the usual method of the G. C. D. In order that there may be a common root a certain condition must be satisfied.

Thus one by one we obtain the values of $\Delta_{10}, \Delta_{20}, \cdots, \Delta_{n0}$, and when these are substituted in the equations of the set (A′) the required conditions are obtained.

The cubic form may be treated in a similar way. In the case of the ternary cubic we must determine two conditions and in the case of the quaternary seven.

A theorem analogous to the one on page 151 obviously exists for the quartic case.